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CONSERVATIVE METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH A CONSERVED QUANTITY

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Abstract. This paper proposes a novel conservative method for the numerical approximation of general stochastic differential equations in the Stratonovich sense with a conserved quantity. We show that the mean-square order of the method is 1 if noises are commutative and that the weak order is 1 in the general case. Since the proposed method may need the computation of a deterministic integral, we analyse the effect of the use of quadrature formulas on the convergence orders. Furthermore, based on the splitting technique of stochastic vector fields, we construct conservative composition methods with similar orders as the above method. Finally, numerical experiments are presented to support our theoretical results.

Key words. Stochastic differential equations, invariants, conservative methods, stochastic geometric numerical integration, quadrature formula, splitting technique, mean-square convergence order, weak convergence order

1. Introduction

In this paper, we consider general d-dimensional autonomous stochastic differential equations (SDE) in the Stratonovich sense

$$\text{(1)}\qquad \mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + \sum_{r=1}^{m} g_r(X(t)) \circ \mathrm{d}W_r(t), \quad 0 \le t \le T, \quad X(0) = X_0,$$

where $W_r(t)$, $r = 1, \cdots, m$ are m independent one-dimensional Brownian motions, defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$. The initial value X_0 is $\mathcal{F}_{t_0}$-measurable with $E|X_0|^2 < \infty$. Here, $f\colon \mathbb{R}^d \to \mathbb{R}^d$ and $g_r\colon \mathbb{R}^d \to \mathbb{R}^d$ are such that the above problem possesses a unique solution. The studies of SDE (1) have drawn dramatic attentions due to its applications in physics, engineering, economics, etc., concerning the effects of random-phenomena. Furthermore, we will assume that equation (1) possesses a scalar conserved quantity $I(x)$, which means that $\mathrm{d}I(X(t)) = 0$ along the exact solution $X(t)$ of (1), e. g. see [2, 6, 7, 9, 16] and references therein for the applications and studies of conservative SDE. Our aim is to derive and analyse numerical methods for (1) preserving this conserved quantity.

Finding numerical solutions of stochastic differential equations is an active ongoing research area, see the review paper [4], the monographs [10, 15] and references therein for instance. Further, it is important to design numerical schemes which preserve the properties of the original problems as much as possible. References [1, 5, 11, 13, 14, 19, 22, 23, 26], without being exhaustive, show general improvements of these so-called geometric numerical methods over more traditional numerical schemes such as Euler-Maruyama's method or the Milstein scheme.

Concerning our problem (1) with a conserved quantity, [16] develops a method to derive conserved quantities from symmetry of SDEs in Stratonovich sense. Further, [17] proposes an energy-preserving method for stochastic Hamiltonian dynamical systems and presents the local error order of the method. The recent work [6]

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proposes a new energy-preserving scheme for stochastic Poisson systems with non-canonical structure matrix and shows that the mean-square convergence order of the scheme is 1. For general SDEs driven by one-dimensional Brownian motion in Stratonovich sense, the authors of [9] propose two conservative methods by means of the skew gradient form of the original SDEs (see below for more details). They also prove that these two methods are convergent with accuracy 1 in the mean-square sense. Based on these two last references, we propose new conservative numerical methods for general stochastic differential equations with a conserved quantity in the present paper.

Since the problem of computing expectations of functionals of solutions to SDEs appears in many applications [25], for example: in finance [20], in random mechanics [24], or in bio-chemistry [8]; we will not only derive the mean-square, but also weak convergence orders of new invariant-preserving numerical methods. Comparing our scheme with the Milstein method, we prove that the mean-square convergence order of our method is 1 under the condition of commutative noise. Furthermore, without assuming any commutativity condition, we show that the weak convergence order of our method is 1. Since the proposed method may need the computation of a deterministic integral, we will also analyse the effect of the use of quadrature formulas on convergence orders. We will show that if the order of a quadrature formula is greater than 2, the mean-square and weak orders of our method remains 1. Based on the splitting technique of stochastic vector fields, we derive new invariant-preserving composition methods of mean-square order one (in the commutative case) and weak order one.

This paper is organized as follows. Section 2 presents the skew gradient form of the problem and derive the proposed invariant-preserving scheme. Properties of the numerical scheme are analyzed in Section 3. The effects of quadrature formula on the mean-square and weak convergence orders and on the discrete conserved quantity are investigated in Section 4. Section 5 deals with the splitting technique of stochastic vector field. Finally, numerical examples are presented to support the theoretical analysis of the previous sections in Section 6.

In the sequel, we will make use of the following notations.

- $|x|$ is the Euclidean norm of a vector x or the induced norm for a matrix.
- We use superscript indices to denote components of a vector or a matrix.
- Partial derivatives are denoted $\partial_i := \dfrac{\partial}{\partial x^i}$ and $\partial_{ij} := \dfrac{\partial^2}{\partial x^i \partial x^j}$ etc.
- $C_b^k(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ is the space of k times continuously differentiable functions $g\colon \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ with uniformly bounded derivatives (up to order $\leq k$).
- $C_P^k(\mathbb{R}^d, \mathbb{R})$ denotes the space of all k times continuously differentiable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ with polynomial growth, i. e., there exists a constant $C > 0$ and $r \in \mathbb{N}$, such that $|\partial^j f(x)| \leq C(1 + |x|^{2r})$ for all $x \in \mathbb{R}^d$ and any partial derivative of order $j \leq k$.

2. Presentation of the conservative method for skew gradients problems

In this section, we will first present the equivalent skew gradient form of (1) with a conserved quantity I, and then we will define our invariant-preserving numerical method.

The equivalent skew gradient form of (1) is stated below.

Proposition 1 (See Theorem 2.2 in [9] for a one-dimensional Brownian motion)**.** *The d-dimensional system* (1) *with a scalar conserved quantity $I(x)$ is equivalent*