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STRONG BACKWARD ERROR ANALYSIS FOR EULER-MARUYAMA METHOD

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Abstract. Backward error analysis is an important tool to study long time behavior of numerical methods. The main idea of it is to use perturbed equations, namely modified equations, to represent the numerical solutions. Since stochastic backward error analysis has not been well developed so far. This paper investigates the stochastic modified equation and backward error analysis for Euler-Maruyama method with respect to strong convergence are built up. Like deterministic case, stochastic modified equations, expressed as formal series, do not converge in general. But there exists the optimal truncation of the series such that the one step error of the modified equations is sub-exponentially small with respect to time step. Moreover, the result of stochastic backward error analysis is used to study the error growth of the Euler-Maruyama method on Kubo oscillator.

Key words. backward error analysis, modified equations, strong convergence, stochastic numerical integrator.

1. Introduction

Backward error analysis is a powerful numerical analysis technique, when the qualitative behavior of numerical methods is of interest, and when statements over long time intervals are needed [4],[8],[15]. However stochastic backward error analysis is still developing, and lots of challenging problems need to be solved. One of the fundamental problems is the construction of stochastic modified equations (ME) with respect to strong convergence, which is used in backward error analysis to approximate numerical solutions. But, to the best of author's knowledge, there is no literature available for any result on strong backward error analysis. We are concerned in this paper with the construction of strong ME for the Euler-Maruyama method (EM).

For an ordinary differential equation,

$$\dot{X}_t = f(X_t), \tag{1}$$

suppose the first order numerical method $\Psi_h(X)$ with a small time step h provides an approximation to the exact solution. and it is represented in a power series of time step:

$$\Psi_h(X) = X + d_1(X)h + d_2(X)h^2 + \cdots .$$

Let ME be in form of power series of h,

$$\dot{\widetilde{X}}_t = \widetilde{f}(\widetilde{X}_t) = f(\widetilde{X}_t) + \widetilde{f}_1(\widetilde{X}_t)h + \widetilde{f}_2(\widetilde{X}_t)h^2 + \cdots ,$$

such that the the flow of ME $\widetilde{\Phi}_h$ matches with the numerical integrator Ψ_h up to arbitrary high order of h.

It is unfortunate that the power series $\widetilde{f}(\widetilde{X}_t)$ does not converge. But, under some appropriate assumption, there exists the optimal truncation such that the

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difference of the numerical method and the flow of ME is exponentially small with respect to time step, that is

$$||\Psi_h - \widetilde{\Phi}_h|| \leq Ce^{-h_0/h},$$

for some constant C and h_0. The important result allows numerical methods to be interpreted by ME. Lots of application on structure preserving method are reported, see [8][9].

To extend the idea to stochastic setting, let us consider stochastic differential equations (SDE) in the form of Stratonovich integral,

$$dX_t = f_0(X_t)dt + \sum_{r=1}^{m} f_r(X_t) \circ dW_t^r = \sum_{r=0}^{m} f_r(X_t) \circ dW_t^r, \tag{2}$$

where $X_t \in \mathbb{R}^n$, $f_r : \mathbb{R}^n \to \mathbb{R}^n$, $W_t^r, r = 1, \ldots, m$ are independent standard Wiener processes, and t is denoted as W_t^0 for notational convenience. Then we say the numerical scheme $\Psi_h(X_0)$ has strong order k if

$$\sqrt{E[||X_{nh} - \Psi_h^n(X_0)||^2]} \leq Ch^k, \quad 0 < nh \leq T.$$

Sometimes a weaker error is sufficient to use. If the numerical scheme $\Psi_h(X_0)$ satisfies that

$$||E[\phi(X_{nh})] - E[\phi(\Psi_h^n(X_0))]|| \leq Ch^k, \quad 0 < nh \leq T,$$

where $\phi(x)$ belongs to some smooth function spaces. Then we say the numerical method has a weak order k.

Shardlow [16] made an attempt by considering the perturbed functions $\widetilde{f}_r$ in stochastic ME have the form

$$\widetilde{f}_r = \sum_{i=0}^{N} \widetilde{f}_{r,i} h^i.$$

When the weak error is considered, the construction can only be performed for EM at $N = 2$ with additive noise. For multiplicative noise or higher order, there are too many conditions to determinate the coefficients of ME. In [17] [1] [2], ME with respect to a week convergence are constructed. Moreover, Debussche et al. [5] built up the weak backward error analysis via modified partial differential equations on torus for EM. Kopac [11] extended the approach to Langvin process on $\mathbb{R}^n$.

In this paper, an alternative approach to construct a perturbed function $\widetilde{f}_r$ is proposed for EM,

$$\widetilde{f}_r = \sum_{\alpha} \widetilde{f}_{r,\alpha} J_{\alpha,t},$$

where $J_{\alpha,t}$ are multiple Stratonovich integrals. Moreover, we prove that there exists the optimal truncation such that

$$\sqrt{E[||\widetilde{\Phi}_{h,N} - \Psi_h||^2]} \leq Ce^{-h_0/(h^{\frac{1}{3}})}.$$

By using this result, the error growth of EM on the Kubo oscillator is investigated.

We emphasize that the proposed modified equations works for SDE with multiplicative noise. The proof given in the paper is different with those provided for ordinary differential equations [9]. We consider the implementation of EM on Kubo oscillator and discuss the error growth of it by using the stochastic backward error analysis result.

The paper is organized as follows. In the next section, we studied the product and second moments of multiple Stratonovich integrals. Then, we introduce the assumption that we need. Section 3 presents the construction of ME. Estimation of