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MULTIPLE NONLINEAR EIGENVALUES OF SMOOTH RANK-DEFICIENT MATRICES

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Abstract. A smooth block LU factorization, coupled with Newton's method, is used to compute multiple nonlinear eigenvalues of smooth rank-deficient matrix functions $A(\lambda)$. We provide conditions for such factorizations to exist and show that the algorithm for the computation of multiple nonlinear eigenvalues converges quadratically, and is more efficient than one using QR factorizations. A possible approach for cubic convergence is also discussed. Several numerical examples are given for general and random nonlinear matrix functions $A(\lambda)$.

Key words. Smooth factorizations, multiple nonlinear eigenvalues.

1. Introduction

The factorization of matrix functions $A(\lambda)$, where λ is a parameter in general complex, has received great attention due to its importance and applications in several areas including linear algebra, matrix computations and dynamical systems. Smooth factorizations in particular [3], where the factors are as smooth as $A(\lambda)$, are a central tool for the computation of heteroclinic and homoclinic orbits in dynamical systems [4], [8].

We say λ is a nonlinear eigenvalue of $A(\lambda)$ if it satisfies

$$A(\lambda)v = 0. \tag{1}$$

The vector $v \neq 0$ is called the corresponding nonlinear eigenvector. In the very special case when $A(\lambda) = B - \lambda I$, for some constant matrix B, then λ and v are just the ordinary (linear) eigenvalue and eigenvector respectively. Nonlinear eigenvalues come from a long list of applications including the dynamical analysis of structures, singularities in elastic materials, and acoustic emissions of high speed trains. For an extensive list of problems related to nonlinear eigenvalues, see [1].

There is reliable software (e.g. MATLAB) that computes such nonlinear eigenvalues for the case when $A(\lambda)$ is a polynomial function. Some algorithms for the cases where $A(\lambda)$ is a general, large, and sparse matrix function have also been developed [11]. One of the main ideas of these methods is to linearize the problem in order to apply linear tools (such as generalized Schur factorizations) to the linearized system and preserve the structure of the original problem.

We explore the idea of computing multiple nonlinear eigenvalues for general, rank-deficient, and smooth matrix functions $A(\lambda)$ via rank-revealing LU factorizations coupled with Newton's method. We assume that $A(\lambda)$ is relatively small and dense. We first introduce the necessary theory to make sure that such factorizations involve smooth factors; we then develop an algorithm, study its convergence properties, and explore a higher order of convergence. A similar approach was considered in [7] using the classical QR factorization, and we take care of comparing their

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corresponding efficiencies. A generalization of this QR approach to compute multiple nonlinear eigenvalues of matrix functions $A(\alpha, \lambda)$, where λ is the eigenvalue parameter, is considered in [2].

2. LU Factorization of a smooth nonsingular matrix

Lemma 1. *All full rank square matrices $A(\lambda) \in C^1$ with nonsingular leading principal submatrices have a unique $L(\lambda)U(\lambda)$ factorization, where $L(\lambda) \in C^1$ is unit lower triangular, and $U(\lambda) \in C^1$ is upper triangular.*

Proof. Let $A(0) = L(0)U(0)$, with $L(0)$ unit lower triangular and $U(0)$ upper triangular. If the sought factorization is feasible, let $A(\lambda) = L(\lambda)U(\lambda)$. Taking the derivative of $A(\lambda)$ and solving for $U'(\lambda)$, we get:

$$U'(\lambda) = L^{-1}(\lambda)A'(\lambda) - L^{-1}(\lambda)L'(\lambda)U(\lambda). \tag{2}$$

Since $U'(\lambda)$ must be upper triangular,

$$\begin{aligned}
(L^{-1}(\lambda)A'(\lambda))_{i,1} &= (L^{-1}(\lambda)L'(\lambda))_{i,1}U(\lambda)_{1,1}, \quad i \geq 2\\
(L^{-1}(\lambda)A'(\lambda))_{i,2} &= (L^{-1}(\lambda)L'(\lambda))_{i,1}U(\lambda)_{1,2} + (L^{-1}(\lambda)L(\lambda))_{i,2}U(\lambda)_{2,2}, \quad i \geq 3.\\
&\vdots
\end{aligned}$$

This system of linear equations is solvable for $B(\lambda) = (L^{-1}(\lambda)L'(\lambda))_{i,j}$ such that $i > j$. Similarly, we have

$$L'(\lambda) = A'(\lambda)U^{-1}(\lambda) - L(\lambda)U'(\lambda)U^{-1}(\lambda). \tag{3}$$

Since $L'(\lambda)$ must be lower triangular,

$$\begin{aligned}
(A'(\lambda)U^{-1}(\lambda))_{1,i} &= L(\lambda)_{1,1}(U'(\lambda)U^{-1}(\lambda))_{1,i}, \quad i \geq 2\\
(A'(\lambda)U^{-1}(\lambda))_{2,i} &= L(\lambda)_{2,1}(U'(\lambda)U^{-1}(\lambda))_{1,i} + L(\lambda)_{2,2}(U'(\lambda)U^{-1}(\lambda))_{2,i}, \quad i \geq 3.\\
&\vdots
\end{aligned}$$

This system of equations is solvable for $C(\lambda) = (U'(\lambda)U^{-1}(\lambda))_{i,j}$ such that $i < j$. The diagonal entries of $U(\lambda)$ are completely determined from $A(\lambda) = L(\lambda)U(\lambda)$. Therefore, the diagonal entries of $C(\lambda)$ depend smoothly on $A(\lambda)$ and $L(\lambda)$. Thus, the system (2), (3) with $A(0) = L(0)U(0)$ is uniquely solvable and provides the sought smooth factorization. □

3. LU factorization for a smooth rank-deficient matrix

Terminology. We say a matrix A has a block LU factorization when $A = LU$ and the matrices L and U satisfy: $L = \begin{bmatrix} L_{1,1} & 0 \\ L_{2,1} & I \end{bmatrix}$, $L_{1,1}$ is a unit lower triangular matrix, $U = \begin{bmatrix} U_{1,1} & U_{1,2} \\ 0 & U_{2,2} \end{bmatrix}$, and $U_{1,1}$ is upper triangular.

Theorem 2. *Let $A(\lambda)$ be an $n \times n$ C^1 matrix function on some $D \subset \mathbb{C}$ such that for some $\lambda_0 \in D$, $A(\lambda_0)$ has rank $n - m$, $m < n$. Assume there are permutation matrices P_1, P_2 such that $P_1A(\lambda_0)P_2$ has a block LU factorization $P_1A(\lambda_0)P_2 = L_0U_0$. Then, there is a neighborhood $N(\lambda_0) \subset D$ such that $P_1A(\lambda)P_2$ has a block LU factorization*

$$P_1A(\lambda)P_2 = L(\lambda)U(\lambda), \ \forall\, \lambda \in N(\lambda_0), \tag{4}$$

with $L(\lambda)$, $U(\lambda) \in C^1(D)$, satisfying that $L(\lambda_0) = L_0$ and $U(\lambda_0) = U_0$.