INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 2011, Number 2-3, Pages 215–230

NUMERICAL SOLUTION OF DYNAMIC PROBLEMS IN COUPLE-STRESSED CONTINUUM ON MULTIPROCESSOR COMPUTER SYSTEMS

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Abstract. Computational algorithm for the solution of 2D and 3D dynamic problems of the Cosserat elasticity theory using multiprocessor computer systems is worked out. The algorithm is based on the two-cyclic splitting method with respect to the spatial variables in combination with the explicit monotone finite-difference ENO–scheme for solving 1D problems adapted for the calculation of discontinuities. Boundary conditions of symmetry, permitting to reduce computations many times, are formulated. Computations of 3D Lamb's problem on the action of concentrated load on the surface of a homogeneous elastic half-space and the problem on the action of concentrated impulsive periodic load are performed. The shock waves of four types (longitudinal, transverse, torsional and rotational waves), characteristic of a couple-stressed elastic medium, are recognized numerically. The oscillations of rotational motion of particles on the wave fronts are also found.

Key words. Cosserat continuum, dynamics, elasticity, shock-capturing method, finite-difference scheme, and parallel computational algorithm.

Introduction

One hundred years celebrated last year from the date of appearance of the first theory of a continuum with internal rotations (the Cosserat continuum), which initiated the development of mechanics of continua with microstructure. The first work by brothers Eugene and Francois Cosserat [1], dedicated to the construction of the couple-stressed elasticity theory, was published in 1909. Further development of this theory refers to the sixties of XX century [2, 3, 4, 5].

Mathematical model of the Cosserat elasticity theory, which takes into account the microstructure of materials, is used to describe the stressed-strained state of composites, powdered, granular, micropolar and liquid-crystal materials. Simplification of this model is a reduced Cosserat model where additional rotational degrees of freedom are taken into account, but the couple stresses are not considered, at first it was suggested in [6]. Current researches on modeling of nanoscale structures show that the Cosserat model results from a limit transition in discrete molecular-dynamic models with an unbounded increase in a number of particles [7, 8, 9]. Therefore, in the near future this model will find wide application.

One of the main factors constraining the analysis of models of non-classical media is a lack of information about the material constants. This, in turn, opposes the adaptation of such models into practice of calculating dynamic characteristics of granular materials. First publications about experimental determination of the elasticity parameters in media with microstructure appeared in 1970 – 1980 years. Parameters for some materials such as bone tissue, highly porous polymer materials and composites were obtained in [10, 11, 12].

Received by the editors October 7, 2010 and, in revised form, May 19, 2011.
2000 *Mathematics Subject Classification.* 35L50, 65Y05, 74S20.

A fundamental difference between the model of the couple-stressed elasticity theory and the classical one is that the former implicitly includes the small parameter that characterizes the size of particles in the microstructure. As a result, in order to obtain correct numerical solutions, computations must be performed on a grid whose meshes are smaller than the characteristic size of particles. To solve 2D and 3D dynamic problems, parallel algorithms can be efficiently used because they make it possible to distribute the computational load between multiple nodes of a cluster. The use of distributed computing allows to do much more fine grids, thereby increasing the accuracy of numerical solutions.

A part of the results of the present paper was reported at the Fifth Conference on Finite Difference Methods: Theory and Applications (June 28 – July 2, 2010, Lozenetz, Bulgaria).

1. Mathematical model

In the model of the Cosserat continuum, along with translational motion which is defined by the velocity vector v, independent rotations of particles with the angular velocity vector ω are considered and, along with the stress tensor σ with nonsymmetric components, the nonsymmetric tensor m of couple stresses is introduced. Complete system of equations of the model consists of the motion equations, the kinematic equations, and the generalized law of the linear elasticity theory:

$$
\begin{aligned}
&\rho\,\dot v = \nabla\cdot\sigma + \rho\,g, \qquad j\,\dot\omega = \nabla\cdot m - 2\,\sigma^a + j\,q,\\
&\dot\Lambda = \nabla v + \omega, \qquad \dot M = \nabla\omega,\\
&\sigma = \lambda\,(\delta : \Lambda^s)\,\delta + 2\,\mu\,\Lambda^s + 2\,\alpha\,\Lambda^a,\\
&m = \beta\,(\delta : M^s)\,\delta + 2\,\gamma\,M^s + 2\,\varepsilon\,M^a.
\end{aligned}
\tag{1}
$$

Here ρ is the material density, g and q are the vectors of body forces and moments, Λ and M are the strain and curvature tensors, j is the product of the moment of inertia of a particle about to the axis passing through its center of gravity by the number of particles in a unit volume, δ is the metric tensor, and λ, μ, α, β, γ, ε are the phenomenological elasticity coefficients for an isotropic medium. The conventional notations and operations of the tensor analysis are used: a colon denotes double convolution of tensors, a dot over a symbol denotes a derivative with respect to time, and superscripts s and a denote the symmetric and antisymmetric tensor components. Where needed, the antisymmetric component is identified with the corresponding vector. In particular, the vector of the tensor $\sigma^a = (\sigma - \sigma^*)/2$ appears in the equations of motion (an asterisk denotes a tensor transposition).

The formula $r = \sqrt{5\,j/(2\,\rho)}$ is valid to estimate the linear parameter of microstructure of a material. It is based on the model approximation about a medium as dense packing of an ensemble of ball-shaped particles of identical radius r.

For 3D case the system of equations (1) in a Cartesian coordinate system has the next expanded form:

$$
\begin{aligned}
&\rho\,\dot v_i = \sigma_{1i,1} + \sigma_{2i,2} + \sigma_{3i,3} + \rho\,g_i,\\
&a_1\,\dot\sigma_{ii} + a_2\,(\dot\sigma_{kk} + \dot\sigma_{ll}) = v_{i,i},\\
&a_3\,\dot\sigma_{ik} + a_4\,\dot\sigma_{ki} = v_{k,i} - \omega_l, \qquad a_4\,\dot\sigma_{ik} + a_3\,\dot\sigma_{ki} = v_{i,k} + \omega_l,\\
&j\,\dot\omega_i = m_{1i,1} + m_{2i,2} + m_{3i,3} + \sigma_{kl} - \sigma_{lk} + j\,q_i,\\
&b_1\,\dot m_{ii} + b_2\,(\dot m_{kk} + \dot m_{ll}) = \omega_{i,i},\\
&b_3\,\dot m_{ik} + b_4\,\dot m_{ki} = \omega_{k,i}, \qquad b_4\,\dot m_{ik} + b_3\,\dot m_{ki} = \omega_{i,k}.
\end{aligned}
\tag{2}
$$