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MONOTONE ITERATES WITH QUADRATIC CONVERGENCE RATE FOR SOLVING SEMILINEAR PARABOLIC PROBLEMS

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Abstract. This paper deals with numerical solving semilinear parabolic problems based on the method of upper and lower solutions. A monotone iterative method with quadratic convergence rate is constructed. The monotone iterative method combines an explicit construction of initial upper and lower solutions and the modified accelerated monotone iterative method. The monotone iterative method leads to the existence-uniqueness theorem. An analysis of convergence rates of the monotone iterative method, based on different stoping tests, is given. Results of numerical experiments are presented, where iteration counts are compared with a monotone iterative method, whose convergence rate is linear.

Key words. semilinear parabolic problem, upper and lower solutions, monotone iterative method, quadratic convergence rate.

1. Introduction

We are interested in monotone iterative methods for solving semilinear parabolic problems in the form

$$\frac{\partial u}{\partial t} - Lu + f(x,t,u) = 0, \quad (x,t) \in \omega \times (0,T], \tag{1}$$

$$u(x,t) = g(x,t), \quad (x,t) \in \partial\omega \times (0,T], \quad u(x,0) = \psi(x), \quad x \in \overline{\omega},$$

where ω is a connected bounded domain in $\mathbb{R}^\kappa$ ($\kappa = 1, 2, \ldots$) with boundary $\partial\omega$. The differential operator L is given by

$$Lu = \sum_{\nu=1}^{\kappa} \frac{\partial}{\partial x_\nu}\left(D(x,t)\frac{\partial u}{\partial x_\nu}\right) + \sum_{\nu=1}^{\kappa} v_\nu(x,t)\frac{\partial u}{\partial x_\nu},$$

where the coefficients of the differential operator are smooth and D is positive in $\overline{\omega} \times [0,T]$. It is also assumed that the functions f, g, and ψ are smooth in their respective domains, and f satisfies the constrain

$$f_u \geq 0, \quad (x,t,u) \in \overline{\omega} \times [0,T] \times (-\infty, \infty). \tag{2}$$

This assumption can always be obtained by a change of variables.

Various reaction-diffusion-convection-type problems in the chemical, physical and engineering sciences are described by problem (1).

In solving such nonlinear problems by the finite difference or finite element methods, the corresponding discrete problem on each discrete time level is usually formulated as a nonlinear system of algebraic equations. A basic mathematical concern of this problem is whether the nonlinear system possesses a solution. This nonlinear system requires some iterative method for the computation of numerical solutions. This leads to the question of convergence of the sequence of iterations. The aim of this paper is to investigate the above questions concerning the existence and

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uniqueness of a solution to the nonlinear system, methods of iterations for the computation of the solution.

The iterative approach presented in this paper is based on the method of upper and lower solutions and associated monotone iterates. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem. The above monotone iterative method is well known and has been widely used for continuous and discrete parabolic boundary value problems. Most of publications on this topic involve monotone iterative schemes whose rate of convergence is of linear rate (cf. [2, 5, 6]). In [7], an accelerated monotone iterative method for solving discrete parabolic boundary value problems is presented. An advantage of this accelerated approach is that it leads to sequences which converge either quadratically or nearly quadratically. In the recent paper [9], a combination of the accelerated monotone iterative method from [7] with monotone Picard iterates is constructed. In [7, 9], the two important points in investigating the monotone iterative method concerning a stopping criterion on each time level and estimates of convergence rates, in the case of solving linear discrete systems on each time level inexactly, were omitted.

In this paper, we extend the accelerated monotone iterative method from [7] to the case when on each time level a nonlinear difference scheme is solved inexactly, and give an analysis of a convergence rate of the modified accelerated monotone iterative method. In [7], it is assumed that a pair of ordered upper and lower solutions is given on each time level, and this pair is used as initial iterates in the accelerated monotone iterative method. Our iterative method combines an explicit construction of initial upper and lower solutions and the modified accelerated monotone iterative method.

The structure of the paper as follows. In Section 2, we introduce a nonlinear difference scheme for the numerical solution of (1). The modified accelerated monotone iterative method is presented in Section 3. The explicit construction of initial upper and lower solutions is incorporated in the monotone iterative method. Quadratic convergent rate of the monotone iterative method is proved. Section 4 deals with existence and uniqueness of the solution to the nonlinear difference scheme. An analysis of convergence rates of the monotone iterative method, based on different stoping tests, is given in Section 5. The final Section 6 presents results of numerical experiments where iteration counts are compared with the monotone iterative method from [2], whose convergence rate is linear.

2. The nonlinear difference scheme

On the domains $\overline{\omega}$ and $[0,T]$, we introduce meshes $\overline{\omega}^h$ and $\overline{\omega}^\tau$, respectively. For solving (1), consider the nonlinear two-level implicit difference scheme

$$\tau_k^{-1}\left[U(p,t_k) - U(p,t_{k-1})\right] + \mathcal{L}^h(p,t_k)U(p,t_k) + f(p,t_k,U) = 0, \tag{3}$$

$$(p,t_k) \in \omega^h \times \omega^\tau,$$

with the boundary and initial conditions

$$U(p,t_k) = g(p,t_k), \quad (p,t_k) \in \partial\omega^h \times \omega^\tau, \quad U(p,0) = \psi(p), \quad p \in \overline{\omega}^h,$$

where $\partial\omega^h$ is the boundary of $\overline{\omega}^h$ and time steps $\tau_k = t_k - t_{k-1}$, $k \geq 1$, $t_0 = 0$. When no confusion arises, we write $f(p,t_k,U(p,t_k)) = f(p,t_k,U)$.