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MULTISCALE ANALYSIS AND COMPUTATION FOR PARABOLIC EQUATIONS WITH RAPIDLY OSCILLATING COEFFICIENTS IN GENERAL DOMAINS

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Abstract. This paper presents the multiscale analysis and computation for parabolic equations with rapidly oscillating coefficients in general domains. The major contributions of this study are twofold. First, we define the boundary layer solution and the convergence rate with $\varepsilon^{1/2}$ for the multiscale asymptotic solutions in general domains. Secondly, a highly accurate computational algorithm is developed. Numerical simulations are then carried out to validate the theoretical results.

Key words. Parabolic equation with rapidly oscillating coefficients, homogenization, multiscale analysis, finite element method.

1. Introduction

We consider the initial-boundary value problems for second order parabolic equations with rapidly oscillating coefficients as follows:

$$(1)\qquad \begin{cases} \dfrac{\partial u^\varepsilon(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\big(a_{ij}^\varepsilon(x,t)\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\big) = f(x,t), \quad (x,t)\in\Omega\times(0,T) \\ u^\varepsilon(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T) \\ u^\varepsilon(x,0) = \bar{u}_0(x), \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded convex polygonal domain with the boundary $\partial\Omega$. $f(x,t)$, $g(x,t)$, $\bar{u}_0(x)$ are known functions. In this study, we consider the following specific cases for the coefficients $a_{ij}^\varepsilon(x,t)$: i.e. $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, and $k = 0,1,2,3$.

Let $\xi = \varepsilon^{-1}x,\ \tau = \varepsilon^{-k}t,\ k = 0,1,2,3$. We make the following assumptions:

$(\mathbf{A_1})$ For $k = 1,2,3$, $a_{ij}(\xi,\tau)$ are 1-periodic and τ_0-periodic in ξ,τ, respectively. For $k = 0$, $a_{ij}(\xi,t)$ are 1-periodic in ξ.

$(\mathbf{A_2})$ $a_{ij} = a_{ji}$, $\gamma_0|\eta|^2 \le a_{ij}(\xi,\tau)\eta_i\eta_j \le \gamma_1|\eta|^2$, $\gamma_0,\gamma_1 > 0$, $\forall(\eta_1,\cdots,\eta_n)\in\mathbb{R}^n$, where γ_0, γ_1 are constants independent of ε.

$(\mathbf{A_3})$ Let $Q = (0,1)^n$ be the reference cell of composite materials with a periodic microstructure, $Q \subset\subset Q'$ and $Q' = (\bigcup_{m=1}^{L} \overline{D}_m)\setminus\partial Q'$. Suppose that the boundaries ∂D_m are $C^{1,\gamma}$ for some $0<\gamma<1$. $a_{ij}^\varepsilon(x,t)\in C^{\mu,\infty}(\overline{D}_m\times(0,T))$, $i,j = 1,2,\cdots,n$ for some constants $0<\mu<1$.

$(\mathbf{A_4})$ $f\in L^2(0,T;L^2(\Omega))$, $g\in L^2(0,T;H^{1/2}(\partial\Omega))$, $\bar{u}_0\in H^1(\Omega)$.

Problem (1) arises frequently in modeling the heat and mass transfer problem in composite materials or porous media (see, e.g., [11]). It involves materials with

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a large number of heterogeneities (inclusions or holes). For homogenization results concerning linear parabolic equations with rapidly oscillating coefficients which depend on the spatial and time variables, we refer to Bensoussan, Lions and Papanicolaou [2] for periodic cases and to Colombini and Spagnolo [7] for the general non-periodic case. For a type of nonlinear parabolic partial differential operators, Pankov [23] and Svanstedt [24] derived the G-convergence and the homogenization results. Zhikov, Kozlov and Oleinik [26] investigated parabolic operators with almost periodic coefficients and presented convergence results for the asymptotic homogenization.

By introducing the cutoff function, Bensoussan, Lions and Papanicolaou (cf. [2]) obtained the strong convergence result without an explicit rate for the first-order corrector of the solution of linear parabolic equations in $L^2(0,T;H^1(\Omega))$. Brahim-Otsmane, Francfort and Murat (cf. [3]) extended this result to $L^2(0,T;W^{1,1}(\Omega))$. Ming and Zhang (cf.[21]) derived the convergence result with an explicit rate $\varepsilon^{1/2}$ for the case $k=0$ under the assumption $u^0\in H^{3,1}(\Omega\times(0,T))$, where $u^0(x,t)$ is the solution of the linear homogenized parabolic equation. Allegretto, Cao and Lin (cf. [1]) investigated the higher-order multiscale method for linear parabolic equations in four specific cases $k=0,1,2,3$, and derived the convergence results with an explicit rate $\varepsilon^{1/2}$ under the assumption $u^0\in H^{s+2,1}(\Omega\times(0,T))$, $s=1,2$. It is well known that, for a bounded polygonal Lipschitz domain Ω, the assumptions $u^0\in H^{s+2,1}(\Omega\times(0,T))$, $s=1,2$ may be invalid. Thus the error estimates in [1] fail. In this study, we present the following two major contributions. First, we define the boundary layer solution and derive the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale asymptotic solutions in a bounded polygonal Lipschitz domain Ω. Secondly, we present a highly accurate computational algorithm.

The remainder of this paper is organized as follows. Section 2 is devoted to the proofs of the main convergence results for the multiscale asymptotic method. In Section 3, we discuss finite element computations and the error estimates for the related problems. In particular, a new computational scheme is proposed to solve the boundary layer solutions numerically. In Section 4, a finite element post-processing technique and a numerical method with high accuracy are presented. Finally, numerical simulations are carried out to validate the theoretical results reported in this paper.

Throughout the paper the Einstein summation convention on repeated indices is adopted. By C we shall denote a positive constant independent of ε.

2. Multiscale Asymptotic Expansions and the Convergence Results

In this section, we first introduce the multiscale asymptotic expansions for problem (1) which has been investigated in [1]. Then we define the boundary layer solutions and derive the convergence results for the modified multiscale asymptotic solutions.

Let $\xi=\varepsilon^{-1}x$, $\tau=\varepsilon^{-k}t$, $k=0,1,2,3$. For the four specific cases $k=0,1,2,3$, following the idea of [1], we define the formal multiscale asymptotic expansions of the solution for problem (1) given by

$$
\begin{aligned}
u_1^{\varepsilon}(x,t)&=u^0(x,t)+\varepsilon N_{\alpha_1}(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}},\\
u_2^{\varepsilon}(x,t)&=u^0(x,t)+\varepsilon N_{\alpha_1}(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_{\alpha_1}}+\varepsilon^2 N_{\alpha_1\alpha_2}(\xi,\tau)\frac{\partial^2 u^0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}},
\end{aligned}
\tag{2}
$$