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Analysis of a Mixed Finite Element Method for Stochastic Cahn-Hilliard Equation with Multiplicative Noise

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Abstract. This paper proposes and analyzes a novel fully discrete finite element scheme with an interpolation operator for stochastic Cahn-Hilliard equations with functional-type noise. The nonlinear term satisfies a one-sided Lipschitz condition and the diffusion term is globally Lipschitz continuous. The novelties of this paper are threefold. Firstly, the L^2-stability (L^∞ in time) and H^2-stability (L^2 in time) are proved for the proposed scheme. The idea is to utilize the special structure of the matrix assembled by the nonlinear term. None of these stability results has been proved for the fully implicit scheme in existing literature due to the difficulty arising from the interaction of the nonlinearity and the multiplicative noise. Secondly, higher moment stability in L^2-norm of the discrete solution is established based on the previous stability results. Thirdly, the Hölder continuity in time for the strong solution is established under the minimum assumption of the strong solution. Based on these findings, the strong convergence in H^{-1}-norm of the discrete solution is discussed. Several numerical experiments including stability and convergence are also presented to validate our theoretical results.

AMS subject classifications: 60H35, 65N12, 65N15, 65N30

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1 Introduction

Consider the following stochastic Cahn-Hilliard (SCH) problem:

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$$du = \Big[-\Delta\Big(\epsilon\Delta u - \frac{1}{\epsilon}f(u)\Big)\Big]dt + \delta g(u)dW_t \quad \text{in } \mathcal{D}_T := \mathcal{D}\times(0,T], \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Big(\epsilon\Delta u - \frac{1}{\epsilon}f(u)\Big) = 0 \quad \text{in } \partial\mathcal{D}_T := \partial\mathcal{D}\times(0,T], \tag{1.2}$$

$$u = u_0 \quad \text{in } \mathcal{D}\times\{0\}, \tag{1.3}$$

where $\mathcal{D}\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded domain, n is the unit outward normal, $\delta>0$ is a positive constant, and W_t is a standard real-valued Wiener process on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t : t\geq 0\},\mathbb{P})$. The analysis in this paper can be generalized to $[L^2(D)]^d$-valued Q-Wiener process. The function f is the derivative of a smooth double equal well potential function F given by

$$F(u) = \frac{1}{4}(u^2-1)^2. \tag{1.4}$$

The diffusion term g is assumed to have zero mean, is globally Lipschitz continuous (1.5), and satisfies the growth condition (1.6), i.e.,

$$|g(a)-g(b)| \leq C|a-b|, \tag{1.5}$$

$$|g(a)|^2 \leq C(1+a^2). \tag{1.6}$$

The SCH problem (1.1)-(1.3) can be rewritten in the following mixed formulation by substituting in the so-called chemical potential $w := -\epsilon\Delta u + \frac{1}{\epsilon}f(u)$:

$$du = \Delta w dt + \delta g(u)dW_t \quad \text{in } \mathcal{D}_T, \tag{1.7}$$

$$w = -\epsilon\Delta u + \frac{1}{\epsilon}f(u) \quad \text{in } \mathcal{D}_T, \tag{1.8}$$

$$\frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\mathcal{D}_T, \tag{1.9}$$

$$u = u_0 \quad \text{on } \mathcal{D}\times\{0\}. \tag{1.10}$$

The mixed formulation will be used to develop the fully discrete finite element scheme in this paper.

The deterministic Cahn-Hilliard equation was originally introduced in [12] to describe phase separation and coarsening processes in a melted alloy. It was proved in [2, 15, 41, 44] that the chemical potential approaches the Hele-Shaw problem as the interactive length ϵ decreases to 0. Numerical justification for this approximation can be found in [24,28,30,31,40,45]. We refer to some other references [17,25–27,33,43,46] about numerical approximation for the Cahn-Hilliard equation and the references therein. For stochastic cases, the Cahn-Hilliard-Cook equation with additive noise (with fixed ϵ) was studied in [14, 32, 35–38, 42]. The well-posedness of the stochastic Cahn-Hilliard equation was discussed in [18, 21, 23] for additive noise and in [5, 13, 39] for multiplicative