Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0012

Vol. **35**, No. 4, pp. 905-937
April 2024

A Characteristic Mapping Method for Vlasov–Poisson with Extreme Resolution Properties

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Received 19 January 2024; Accepted (in revised version) 2 April 2024

Abstract. We propose an efficient semi-Lagrangian characteristic mapping method for solving the one+one-dimensional Vlasov–Poisson equations with high precision on a coarse grid. The flow map is evolved numerically and exponential resolution in linear time is obtained. Global third-order convergence in space and time is shown and conservation properties are assessed. For benchmarking, we consider linear and nonlinear Landau damping and the two-stream instability. We compare the results with a Fourier pseudo-spectral method and results from the literature. The extreme fine-scale resolution features are illustrated showing the method's capabilities to efficiently treat filamentation in fusion plasma simulations.

AMS subject classifications: 35Q83, 65M25, 68W10, 76X05

Key words: Characteristic mapping method, kinetic equations, Vlasov–Poisson, plasma.

1 Introduction

Flows in the general setting transport quantities, e.g. fluids, plasmas, particles or their probability distribution function (PDF), from one place to another. Typically they generate rich mathematical multi-scale structures even from simple analytical initial conditions and require well-adapted numerical methods to solve the underlying governing partial differential equations. In the present work, we focus on the Vlasov–Poisson (VP) system, modeling particle evolution under their self-consistent electric field, and neglecting their

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collisions. We consider the equation in 1d+1d, i.e. in the space of positions and velocities. This transport equation in phase space has numerous applications in plasma and astrophysics [32]. A prominent example is Landau damping in statistical physics of ionized gases.

As a kinetic equation, the VP equation is intrinsically high-dimensional, however, it is especially challenging due to the filamentation of the particle distribution function in phase space. The filamentation is a physical property of the system, that leads to a continuous generation of fine scales with increasing time. Resolving these scales requires computer simulations to accumulate more and more computer storage and resources within time.

To overcome the filamentation problem in numerical simulations two main approaches have been followed: the Lagrangian approach and the semi-Lagrangian approach. These methods shift the evolution of the PDF towards the evolution of its underlying transport structure. In the purely Lagrangian approach, known as the particle-in-cell method (PIC) (see for review [5, 21, 42]), a set of N_p particles is used to statistically approximate the distribution function in velocity space. The particles move according to their equations of motion in an electromagnetic field that is interpolated from a fixed grid to the particle's position. As the PDF is determined statistically the overall result is subject to statistical noise, with the variance decreasing only slowly with $1/\sqrt{N_\mathrm{p}}$. Different approaches for denoising have been proposed, see e.g. [35], among them are wavelet-based density estimation [19, 41].

In order to obtain even more accurate results, while simultaneously respecting the particle's equation of motion, semi-Lagrangian approaches have been developed. This special type of Eulerian method represents the PDF on a fixed grid whereas time is evolved with the help of particle trajectories, known as the characteristic curve. Along these curves, the PDF is conserved and thus constant in time. The PDF is computed by tracing back the trajectory and interpolating its origin back on an Eulerian mesh. The first approach in this direction was already presented in the seventies by Cheng and Knorr [8] and has ever since got continuous attention in research [3, 4, 9–11, 14, 28, 37, 38].

In [38] Sonnendrücker et al. formalized the semi-Lagrangian approach for solving the VP equations by employing spline interpolations at the feet of the characteristics. While this method is renowned for its high precision, it is accompanied by the need to solve a global tri-diagonal system induced by the spline interpolation scheme. To overcome this computational challenge, [4] introduced local interpolation schemes based on Hermite and Lagrange polynomials, which were shown to exhibit high-order convergence results in space and second-order accuracy in time, as documented by [3]. The introduction of local interpolation schemes enhanced the parallelizability and adaptability of the method to unstructured meshes [4]. However, it introduced the requirement to transport gradients as additional fields. Subsequent developments in this direction focused on maintaining positivity and mass conservation, as presented for the backward [9], and the forward semi-Lagrangian method [11].

Another prominent methodology in the semi-Lagrangian setting is built on discon-