Journal of Computational Mathematics
Vol.42, No.4, 2024, 1172–1196.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2301-m2022-0177

A CELL-CENTERED GODUNOV METHOD BASED ON STAGGERED DATA DISTRIBUTION, PART I: ONE-DIMENSIONAL CASE*

Jiayin Zhai
Institute of Applied Physics and Computational Mathematics, LCP, P.O. Box 8009-26, Beijing 100088, China
Email: jiayinzhai@sina.com
Xiao Li
College of Big Data and Internet, Shenzhen Technology University, Shenzhen 518118, China
Email: lixiao@sztu.edu.cn
Zhijun Shen[1)]
Institute of Applied Physics and Computational Mathematics, LCP, P.O. Box 8009-26, Beijing 100088, China
Center for Applied Physics and Technology, HEDPS, Peking University, China
Email: shen_zhijun@iapcm.ac.cn

Abstract

This paper presents a cell-centered Godunov method based on staggered data distribution in Eulerian framework. The motivation is to reduce the intrinsic entropy dissipation of classical Godunov methods in the calculation of an isentropic or rarefaction flow. At the same time, the property of accurate shock capturing is also retained. By analyzing the factors that cause nonphysical entropy in the conventional Godunov methods, we introduce two velocities rather than a single velocity in a cell to reduce kinetic energy dissipation. A series of redistribution strategies are adopted to update subcell quantities in order to improve accuracy. Numerical examples validate that the present method can dramatically reduce nonphysical entropy increase.

Mathematics subject classification: 35Q35, 76N15, 76M12.
Key words: Godunov method, Riemann solver, Rarefaction wave, Nonphysical entropy increase.

1. Introduction

A cell-centered Godunov method always generates numerical entropy increase, even if in isentropic flows. A typical phenomenon is the over-predicting temperature in low internal energy flows. For example, when two fluid streams recede from each other, a rarefying region between them is generated. However, numerical solutions consistently indicate a temperature rise rather than a correct decrease. In general, such temperature anomaly belongs to a kind of entropy error [15, 16]. It appears in the extensive Godunov-type finite volume methods, such as exact or approximate Riemann solvers, and the flux splitting methods [23] in the Eulerian and the Lagrangian frameworks. Solving this problem has theoretical and practical interests, such as multi-dimensional simulation of the strong compression phase in Inertial Confinement Fusion. A possible numerical result is that the compressed air has wrong compression ratio

* Received July 29, 2022 / Revised version received October 31, 2022 / Accepted January 28, 2023 / Published online September 13, 2023 /
1) Corresponding author

with the experimental comparison. Therefore, it is necessary to design one numerical method that can remove the entropy error in the rarefying problems but also retain the property of accurate shock capturing simultaneously.

Many researchers have attempted to solve these kinds of problems. One exploration of reducing such overheating phenomenon is to employ high-order scheme or refine grids, as shown in [1,6]. Regretfully, the excessive increase in temperature does not reduce by refining the mesh size, decreasing the time step [16], or increasing the order of accuracy [11]. For very low internal energy flows, high-order conservative schemes may show less ability to obtain results with sufficient accuracy [11] due to difficulty in maintaining positivity. Historically speaking, the first work to connect the phenomenon with entropy conservation is Tadmor [21]. He developed a semi-discrete entropy-conservative (EC) flux for smooth flows, and a proper dissipation term is added for non-smooth flows. Roe [20] overcame the difficulty of explicit form of the EC flux and proposed an explicit entropy-stable flux for the Eulerian gas dynamics equations. However, their methods are aimed at ensuring consistency of numerical methods and thermodynamics relation, and not at solving the isentropic problem. In a series of works [14–16], Liou found that this temperature increase is solely connected to entropy rise. Furthermore, he regarded that the entropy rise is rooted in the pressure flux in a finite volume formulation and is theoretically inevitable for all existing numerical flux schemes used in the finite volume setting. In order to solve this problem, a point value characteristics method was adopted [15]. Cocchi *et al.* [8] suggested a non-conservative formulation for the energy equation to decrease the nonphysical temperature increase. Clearly, these non-conservative methods are not suited for flows where shock waves and isentropic flows coexist. In the Lagrangian frame, Hui and Kudriakov [12] expressed similar viewpoint with those in [14–16] in an earlier study and proposed a shock-tracking method. Their method can remove many numerical defects in one-dimensional calculations, but can not be extended to multi-dimensional cases due to the introduction of exact Riemann solution. Braeunig [2] and Paulin *et al.* [19] proposed an Enhanced Entropy Behavior method to improve effect of computing rarefaction waves. The main work is to set a switch at a cell interface to distinguish compressible and expansion states. Similar work has been performed by Maire *et al.* [17] with Tadmor's theory, whose numerical scheme adopts a hybrid form to treat compression and rarefaction cases and the setting idea of switch is identical in essence to that in [2]. However, designing a correct criterion itself is a huge challenge for a cell-centered method. In fact, a pure rarefaction wave in physics may have alternative compressive and expansion behaviors at the discrete level. In general, this method does not preserve grid-convergence and thus the entropy error might not decrease with the grid refinement.

In our understanding, there are two factors that might cause nonphysical entropy errors. One factor is the excessive dissipation in the numerical flux, as mentioned above, which has close relation with criterion to cell state. In conventional Godunov methods, the compressible or expansion state of a cell is a combination of all boundaries working simultaneously, and thus it results in difficulty to modify viscosity in a single Riemann solver with local initial inputs. Another is the momentum average in Godunov method, which causes the loss of kinetic energy and further leads to the increase of internal energy [28]. In fact, there are two stages in a Godunov-type scheme to update flow variables: the first stage is evolution procedure of the numerical fluxes across a cell interface and the second is the averaging procedure for the reconstruction of constant state inside each cell. The momentum average in the second stage can induce kinetic energy dissipation. Furthermore, Thornber *et al.* [22] pointed out that the amount of numerical entropy increase is proportional to the velocity jump squared.