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TWO-GRID FINITE ELEMENT METHOD FOR TIME-FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION*

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Abstract

A two-grid finite element method with $L1$ scheme is presented for solving two-dimensional time-fractional nonlinear Schrödinger equation. The finite element solution in the L^{∞}-norm are proved bounded without any time-step size conditions (dependent on spatial-step size). The classical $L1$ scheme is considered in the time direction, and the two-grid finite element method is applied in spatial direction. The optimal order error estimations of the two-grid solution in the L^p-norm is proved without any time-step size conditions. It is shown, both theoretically and numerically, that the coarse space can be extremely coarse, with no loss in the order of accuracy.

Mathematics subject classification: 65M06, 65M12, 65M15, 65M55.
Key words: Time-fractional nonlinear Schrödinger equation, Two-grid finite element method, The $L1$ scheme.

1. Introduction

We consider the initial-boundary value problem of the two-dimensional time-fractional nonlinear Schrödinger (NLS) equation

$$\begin{cases} \mathrm{i}\,{}_0^C D_t^{\alpha} u + \Delta u + f(|u|^2)u = 0 & \text{in } \varOmega \times (0,T], \\ u = 0 & \text{on } \partial\varOmega \times (0,T], \\ u = u_0 & \text{on } \varOmega \times \{0\} \end{cases} \tag{1.1}$$

in a convex polygonal domain $\varOmega \subset \mathbb{R}^2$ with boundary $\partial\varOmega$, where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit and $u : \varOmega \to \mathbb{C}$ is the complex-valued unknown solution, and $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function and $f \in C^2(\mathbb{R})$. ${}_0^C D_t^{\alpha} u$ is defined by the following Caputo fractional derivative of order α:

$${}_0^C D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha}}, \quad 0 < \alpha < 1, \tag{1.2}$$

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where $\Gamma(\cdot)$ denotes the usual Gamma function. According to [35, Theorems 3.1-3.2], it can easily see that the Eq. (1.1) admits a unique solution.

In recent years, time-fractional NLS equations have been used to describe many physical phenomena, see [23, 32, 33]. For example, Naber [33] obtained the time-fractional Schrödinger equation to describe non-Markovian evolution of a free particle in quantum physics. When α tends to 1, the fractional derivative ${}_0^C D_t^\alpha u$ would converge to the first-order derivative $\partial u/\partial t$, and thus (1.1) reproduces the standard Schrödinger equation [26].

In the past decades, developing effective numerical methods and rigorous numerical analysis for the time-fractional PDEs and other PDEs have been a hot research spot [8, 12–16, 24, 25, 30, 31, 34, 44–46]. Recently, many works have been used to develop effective numerical methods for solving time-fractional problems. For the time fractional derivative, the time fractional differential equations was generally transformed into integro-differential equations for analysis. However, in recent years, some direct methods were used to approximate the time-fractional derivative due to the numerical schemes are implemented much more easily [4, 25, 29]. One of the most commonly used direct methods is the so-called $L1$ scheme, which can be viewed as a piecewise linear approximation to the fractional derivative. Due to need evaluate the special sum of all the solutions in the previous time levels for the time-fractional PDEs, many authors work on the fast algorithms. Gao *et al.* [11] developed a high-order formula called $L1-2$ formula which had the temporal convergence order $\mathcal{O}(\tau^{3-\alpha})$ at time $t^j\,(j \geq 2)$ but $\mathcal{O}(\tau^{2-\alpha})$ at t^1 for the time fractional sub-diffusion equation. Alikhanov [1] constructed a formula called $L2-1_\sigma$ formula which has temporal convergence order $\mathcal{O}(\tau^{3-\alpha})$ for time $t^j\,(j \geq 1)$ for the time fractional diffusion equation. Later, Yan *et al.* [43] developed a high-order formula called $FL1-2_\sigma$ formula for the time fractional diffusion equation. Gu *et al.* [13,15,46] considered the fast (or parallel) method for time-fractional PDEs, which can reduce the computational cost from both the spatial and temporal discretizations.

Some works of $L1$ schemes have been done for nonlinear time-fractional NLS equations. For example, the $L1$ schemes with different approximations in a spatial direction, such as finite difference method [4], finite element method [29, 39], spectral collocation methods [2, 3] and discontinuous Galerkin method [40]. Recently, Li *et al.* [29] developed linearized $L1$ schemes for the time-fractional NLS equations and proved unconditionally optimal error estimates. Chen *et al.* [4] proposed two linearized compact alternating direction implicit (ADI) methods for solving the two-dimensional time-fractional NLS equations.

Because the resulting system of time-fractional NLS equation is a nonlinear system, an iterative procedure should be used to solve it, which needs a lot of computational costs. As we known, the two-grid finite element method was introduced by Xu [41, 42] to solve nonlinear elliptic equations efficiently. Later, it has attracted many scholars to do research on two-grid techniques with the finite element methods, mixed element methods, and discontinuous Galerkin methods, such as nonlinear parabolic equations [6, 7, 9], miscible displacement problems [5, 21, 22], the Schrödinger equations [17–20, 37, 38]. To avoid time-step constraints and coarse-fine mesh ratio which is too close to $1\,(H = \mathcal{O}(h^r),\ r$ too close to 1), we need carry out optimal order L^p error estimates of finite element solutions and prove L^∞ bound of finite element solutions. Recently, we proved a L^p norm error estimate for the implicit Crank-Nicolson finite element two-grid method without mesh-ratio conditions for NLS equation with wave operator [20]. Li *et al.* [38] applied two-grid finite element method to solve the nonlinear Schrödinger equation, which deduced unconditional boundedness of the numerical solution by mathematical induction method and unconditional superclose result with order $\mathcal{O}(h^2 + \tau)$ in