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A TRUST-REGION METHOD FOR SOLVING TRUNCATED COMPLEX SINGULAR VALUE DECOMPOSITION*

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Abstract

The truncated singular value decomposition has been widely used in many areas of science including engineering, and statistics, etc. In this paper, the original truncated complex singular value decomposition problem is formulated as a Riemannian optimization problem on a product of two complex Stiefel manifolds, a practical algorithm based on the generic Riemannian trust-region method of Absil *et al.* is presented to solve the underlying problem, which enjoys the global convergence and local superlinear convergence rate. Numerical experiments are provided to illustrate the efficiency of the proposed method. Comparisons with some classical Riemannian gradient-type methods, the existing Riemannian version of limited-memory BFGS algorithms in the MATLAB toolbox Manopt and the Riemannian manifold optimization library ROPTLIB, and some latest infeasible methods for solving manifold optimization problems, are also provided to show the merits of the proposed approach.

Mathematics subject classification: 15A24, 15A57, 65F10, 65F30.
Key words: Truncated singular value decomposition, Riemannian optimization, Trust-region method.

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1. Introduction

In linear algebra, the singular value decomposition (SVD) of a complex matrix $A \in \mathbb{C}^{m\times n}$ is a factorization of the form $U\Sigma V^H$, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices, $\Sigma \in \mathbb{C}^{m\times n}$ is a rectangular diagonal matrix with non-negative real numbers on the diagonal. The diagonal entries $\sigma_i = \Sigma_{ii}$ of Σ are known as the singular values of A. The number of non-zero singular values is equal to the rank of A. The columns of U and the columns of V are called the left-singular vectors and right-singular vectors of A, respectively. The truncated singular value decomposition (TSVD) is a kind of reduced SVDs, in which only the p ($p \le \mathrm{rank}(A)$) column vectors of U and p row vectors of V^H corresponding to the p largest singular values Σ_p are calculated. The rest of the matrix is discarded. This can be much quicker and more economical than the compact SVD if $p \ll \mathrm{rank}(A)$. The matrix U_p is thus $m \times p$, Σ_p is $p \times p$ diagonal, and V_p is $n \times p$. Of course the truncated SVD is no longer an exact decomposition of the original matrix A, but the approximate matrix $\tilde{A} = U_p\Sigma_p V_p^H$ is in a very useful sense the closest approximation to A that can be achieved by a matrix of rank p. To find p ($\le \mathrm{rank}(A)$) left and right singular vectors associated with the p largest singular values of a complex matrix $A \in \mathbb{C}^{m\times n}$, a closely related problem is to solve the following optimization problem [3,27,29–31]:

$$\begin{aligned} &\text{maximize} \quad \mathrm{Tr}(U^H AV\Theta), \\ &\text{subject to} \quad U \in \mathbb{C}^{m\times p}, \quad V \in \mathbb{C}^{n\times p}, \quad U^H U = I_p, \quad V^H V = I_p, \end{aligned} \tag{1.1}$$

where $\Theta = \mathrm{diag}(\mu_1, \cdots, \mu_p)$, with $\mu_1 > \cdots > \mu_p > 0$ arbitrary and $1 \le p \le \mathrm{rank}(A)$, and where I_p is the identity matrix of order p. Here, the arbitrary diagonal matrix Θ with positive diagonal elements in descending order plays a role in appropriately ordering the columns of U and V ([29, Chapter 5] and [30]). In (1.1), the notation $\mathrm{Tr}(\cdot)$ denotes the real part of the trace of a given complex square matrix, which corresponds to the inner product for complex vector space, i.e., in $\mathbb{C}^{p\times q}$, we define the inner product

$$\langle M, N\rangle := \mathrm{Tr}(M^H N) = \mathrm{trace}\left(\mathrm{Re}(M)^T \mathrm{Re}(N)\right) + \mathrm{trace}\left(\mathrm{Im}(M)^T \mathrm{Im}(N)\right)$$

for all $M, N \in \mathbb{C}^{p\times q}$, where $\mathrm{Re}(M)$ and $\mathrm{Im}(M)$ denote the real and imaginary parts of M, respectively. Then $\mathbb{C}^{p\times q}$ is a Hilbert inner product space and the norm of a complex matrix M generated by this inner product is the matrix Frobenius norm given by $\|M\|^2 = \mathrm{Tr}(M^H M)$. A global optimal solution to the problem (1.1) provides a collection of p dominant left and right singular vectors of A. Let (U^*, V^*) be an optimal solution to the problem (1.1). Then, the j-th columns of U^* and V^* are the left and right singular vectors of A associated with the j-th dominant singular value, respectively [3, 27, 29–31]. In addition, the p largest singular values $\sigma_1 \ge \cdots \ge \sigma_p$ can be calculated through the formula ${U^*}^H AV^* = \mathrm{diag}(\sigma_1, \cdots, \sigma_p)$.

Mathematical applications of the TSVD include computing the matrix low-rank approximation [11], analysing the ill-posed models [19] and solving the ill-conditioned system of algebraic equation [12]. In the ill-posed models, the ill conditioned of the model is mainly reflected in the amplification of the small singular values of the coefficient matrix on the parameters and their variances. The basic idea of the truncated singular value method is to cut off these small singular values and reconstruct the coefficient matrix to weaken the ill-posedness of the model. The TSVD is also extremely useful in all areas of science, engineering, and statistics, such as bioluminescence tomography (BLT) [40] and through-the-wall microwave imaging (TWI) [16]. In BLT, TSVD regularization method is applied to solving BLT inverse problem with the source