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# Upper Bound of the Number of Zeros for Abelian Integrals in a Kind of Quadratic Reversible Centers of Genus One*

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**Abstract** By using the methods of Picard-Fuchs equation and Riccati equation, we study the upper bound of the number of zeros for Abelian integrals in a kind of quadratic reversible centers of genus one under polynomial perturbations of degree $n$. We obtain that the upper bound is $7[(n-3)/2]+5$ when $n \geq 5$, 8 when $n=4$, 5 when $n=3$, 4 when $n=2$, and 0 when $n=1$ or $n=0$, which linearly depends on $n$.

**Keywords** Abelian integral, quadratic reversible center, weakened Hilbert's 16th problem, Picard-Fuchs equation, Riccati equation

**MSC(2010)** 34C07, 34C08, 37G15.

## 1. Introduction

The last part of Hilbert's 16th problem is the discussion about the number and relative positions of limit cycles in planar dynamic systems. This problem is so complex and difficult that even the simplest nonlinear case, $n=2$, is not completely solved. In 1977, V.I. Arnold proposed a weakened form of this problem, which is to study maximum of the number of isolated zeros of the Abelian integral for the system. The problem is known as the weakened Hilbert's 16th problem [1], which is one of the hot topics in current research. So far, most of the results about determining the upper bound of the number of isolated zeros of Abelian integrals are related to perturbed Hamiltonian systems. The study of perturbed integrable non-Hamiltonian systems is more difficult than disturbance Hamiltonian systems. More details can be found in the review article [13] and the books [2, 4].

The literature [12] lists all quadratic integrable non-Hamiltonian systems, because quadratic reversible systems have good functional properties. Therefore, they have been widely studied [15–17]. For quadratic reversible centers of genus one, there are essentially 22 cases in the classification in [3], namely $(r1)$–$(r22)$. The linear dependance of case $(r1)$ was studied in [18]; cases $(r3)$–$(r6)$ were studied in [14];

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cases $(r9), (r13), (r17)$ and $(r19)$ were studied in [10]; cases $(r11), (r16), (r18)$ and $(r20)$ were studied in [9]; cases $(r12)$ and $(r21)$ were studied in [8]; case $(r7)$ was studied in [7]; case $(r10)$ was studied in [6]; and $(r22)$ was studied in [5]. All of these upper bounds linearly depend on $n$. In order to thoroughly study these problems, we consider a special case in $(r5)$. Namely, when $(a,b) = (4,2)$, we can get the special case $(sr5)$ as follows:

$$(sr5)\quad \begin{aligned} &\dot{x} = -xy, \quad \dot{y} = -2y^2 + \frac{1}{2^4}x^2 - \frac{1}{2^4}x, \\ &H(x,y) = x^{-4}\left(\frac{1}{2}y^2 - \frac{1}{2^5}x^2 + \frac{1}{3\times 2^4}x\right) = h, \quad h \in \left(-\frac{1}{3\times 2^5}, 0\right), \end{aligned} \tag{1.1}$$

with the integrating factor $N(x,y) = x^{-5}$.

$(sr5)$ is an integrable non-Hamiltonian quadratic system, which has a center $(1,0)$, a family of periodic orbits $\{\Gamma_h\}$ $(-1/(3\times 2^5) < h < 0)$, and an integral curve $x = 0$ (see Figure 1).

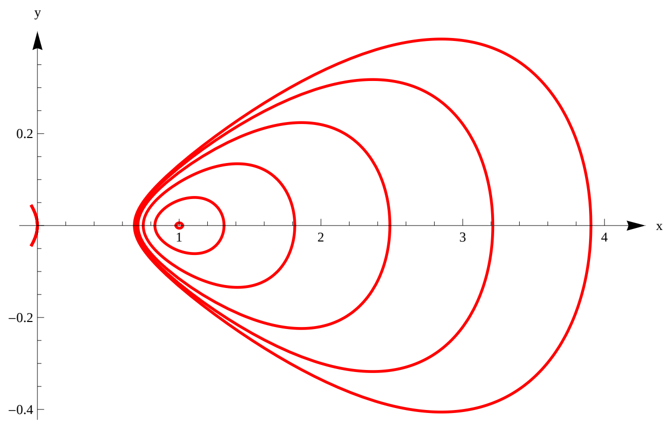


**Figure 1.** The periodic orbits of the system $(sr5)$

We consider the perturbation system of $(sr5)$, whose form is as follows:

$$\dot{x} = -xy + \varepsilon p(x,y), \quad \dot{y} = -2y^2 + \frac{1}{2^4}x^2 - \frac{1}{2^4}x + \varepsilon q(x,y), \tag{1.2}$$

where $\varepsilon$ is a real parameter, and $0 < \varepsilon \ll 1$, $p(x,y)$ and $q(x,y)$ are polynomials of $x$ and $y$, and $\max\{\deg(p(x,y)), \deg(q(x,y))\} = n$.

In this paper, we study the upper bound of the number of zeros for Abelian integrals of (1.2) by using the methods of Picard-Fuchs equation and Riccati equation [11]. Our main result is the following Theorem 1.1.

**Theorem 1.1.** *For the system (1.2), the upper bound for the number of zeros of the Abelian integral*

$$A(h) = \oint_{\Gamma_h} N(x,y)[q(x,y)dx - p(x,y)dy], \quad h \in \Omega, \tag{1.3}$$

*where $\Gamma_h$ are periodic orbits of system (1.1) and are defined on the maximum open interval $\Omega = (h_1, h_2)$, which linearly depends on $n$. Concretely, the upper bound is $7[(n-3)/2] + 5$ when $n \geq 5$, $8$ when $n = 4$, $5$ when $n = 3$, $4$ when $n = 2$, and $0$ when $n = 1$ or $n = 0$.*