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Traveling Wave Solutions in a Chemotaxis Model with Two Chemoattractants*

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Abstract In this work, we investigate the existence and non-existence of traveling wave solutions for a chemotaxis model with two chemoattractants. To prove our main results, we apply the dynamical systems theory by constructing a positively invariant set in the four-dimensional space. Particularly, we analyze the monotonicity of traveling wave solutions.

Keywords Chemotaxis model, traveling wave solutions, dynamical systems theory

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1. Introduction

Our purpose in this work is to investigate the traveling wave solutions of the following chemotaxis model with two chemoattractants,

$$\begin{cases} u_t = u_{xx} - (\chi_1(v)uv_x)_x - (\chi_2(w)uw_x)_x + ru(1-u), \\ v_t = v - \alpha u, \\ w_t = w - \beta u, \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}$ and $t \geq 0$. Here $u(x,t)$ represents the density of cell population, while $v(x,t)$ and $w(x,t)$ represent the densities of chemical concentrations of two different chemicals. The parameter $r > 0$ denotes the rate of logistic cell growth, while $\alpha > 0$ and $\beta > 0$ mean that the cells consume the chemoattractants. The chemotactic sensitivity $\chi_i(\cdot)$ $(i = 1, 2)$ describes the measure of the strength of chemotaxis and is referred as the Chemotactic Coefficient. The cells move towards where the concentration of chemical v and w increase. This motion is represented by $-(\chi_1(v)uv_x)_x$ and $-(\chi_2(w)uw_x)_x$ respectively. To simplify the calculation, we consider that the chemical growth rate of v or w is 1.

In recent years, many experts have focused their attention on the research of biomathematics. Scholars have constructed a series of biological mathematical models based on the behavior of individual organisms seeking benefits and avoiding

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harm, and have provided objective and reasonable explanations and predictions for many phenomena in biology by studying relevant partial differential equations.

One such phenomenon is chemotaxis, which causes species to move in specific directions in response to the attraction of certain chemical signals [1]. This chemotaxis leads to the formation of diverse patterns in nature, thereby creating a rich and colorful world. The reaction diffusion equation system proposed by Turing [2] in 1952 successfully explained the mechanism of speckle phenomenon, also known as the Turing mode. Mathematically speaking, when the parameters change, the stability of the constant equilibrium solution changes, from stable to unstable, and the process of generating non-homogeneous non-constant equilibrium solutions in space is called Turing mode. The mechanism of chemotaxis has been widely applied in daily life, such as trapping and killing pests, infecting bacteria, cultivating microorganisms, and treating wounds. Through theoretical and experimental observations, the morphological generation phenomena of chemotaxis exhibit a rich and colorful structure, including aggregation, finite time explosion, spot patterns, spikes, stripes, rings, etc. Due to the different principles and mechanisms of chemotaxis, simulation of specific systems, and mathematical explanations, a large number of different forms of chemotaxis models have emerged.

Keller and Segel [3] established a chemotaxis model for the first time in the 1970s.

$$\begin{cases} b_t = (\mu(s)b_x)_x - (b\chi_s s_x)_x, \\ s_t = Ds_{xx} - bk(s), \end{cases} \tag{1.2}$$

which was proposed to explicate the phenomenon of aggregation observed in the celebrated experiment of Adler [4,5] in the cellular slime mold Dictyostelium. They showed that the model (1.2) can reproduce the traveling bands whose speeds are consistent with Adler's experimental observation in [7]. After Keller-Segel's work, traveling wave solutions of the chemotaxis models have been widely studied by many other scholars; see [6–15] and the reference therein.

Motivated by Li [14], we consider the chemotaxis model (1.1) with two chemoattractants which no one has studied before. We prove the existence and non-existence of traveling wave solutions by using the dynamical systems theory. Firstly, system (1.1) is transformed into an ODE system. Next, the existence of traveling wave solutions connecting two different equilibria is equivalent to the existence of heteroclinic orbits of the transformed ODE system. Then we construct a positively invariant set of the corresponding ODE system in the four-dimensional phase space that guarantees the existence of the desired heteroclinic orbits.

Our main results for the existence of traveling wave solutions are under the assumptions,

(H1) $0 \le \chi_1(v) \le k_1$ and $\chi_1(\alpha) = 0$,

(H2) $0 \le \chi_2(w) \le k_2$ and $\chi_2(\beta) = 0$,

(H3) $r > \alpha k_1 + \beta k_2$.

where k_i $(i = 1, 2)$ is a constant. Now we present the results on the existence and non-existence of traveling wave solutions.

Theorem 1.1. *Let (H1)-(H3) hold. There exists a minimal speed* $c_* > 0$ *such that for each* $c \ge c^* > 0$ *the system* (1.1) *has a traveling wave solution* $(u(x,t),\ v(x,t),$