Journal of Nonlinear Modeling and Analysis
Volume 6, Number 1, March 2024, 133–141
http://jnma-online.com
DOI:10.12150/jnma.2024.133

Painlevé Analysis and Auto-Bäcklund Transformation for a General Variable Coefficient Burgers Equation with Linear Damping Term*

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Abstract This paper investigates a general variable coefficient (gVC) Burgers equation with linear damping term. We derive the Painlevé property of the equation under certain constraint condition of the coefficients. Then we obtain an auto-Bäcklund transformation of this equation in terms of the Painlevé property. Finally, we find a large number of new explicit exact solutions of the equation. Especially, infinite explicit exact singular wave solutions are obtained for the first time. It is worth noting that these singular wave solutions will blow up on some lines or curves in the (x, t) plane. These facts reflect the complexity of the structure of the solution of the gVC Burgers equation with linear damping term. It also reflects the complexity of nonlinear wave propagation in fluid from one aspect.

Keywords Painlevé property, auto-Bäcklund transformation, a gVC Burgers equation with linear damping term, exact solutions

MSC(2010) 35A30,35K55,35K57, 35Q53, 37K35.

1. Introduction

The well-known Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0, \tag{1.1}$$

as the simplest nonlinear model in fluid dynamics balances the effect of nonlinear convection and the linear diffusion [5]. It was originally derived to describe the propagation of nonlinear waves in dissipative media, where $\nu(> 0)$ is the kinematic viscosity, and $u(x, t)$ represents the fluid velocity field. It plays an important role in explaining two fundamental effects characteristic of any turbulence: the nonlinear redistribution of energy over the spectrum and the action of viscosity in small scales. For the classical Burgers equation, a large number of literatures have discussed its Painlevé integrability, Bäcklund transformation and other mathematical physics properties [1, 4, 9, 10, 24].

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*The first author was supported by National Natural Science Foundation of China (11871172).

In the real processes, the influence of damping is inevitable. It even plays an important role in some problems. The Burgers equation with a linear damping which describes the plane motion of a continuous medium for which the constitutive relation for the stress contains a large linear term proportional to the strain, a small term which is quadratic in the strain, and a small dissipative term proportional to the strain-rate, has been studied for single hump conditions by singular perturbation approach by Lardner and Arya [11]. The N-wave solutions for this equation have been obtained by Sachdev and Joseph [18]. Vaganan and Kumaran [20] discussed similarity solutions of the Burgers equation with linear damping and obtained a trivial solution of the Burgers equation with linear damping by Lie's group analysis method. According to a relationship between the solutions of the damped Burgers equation and the cylindrical Burgers equation obtained by Sachdev and Vaganan [19], they also obtained a solution of the cylindrical Burgers equation. Peng and Chen [15] obtained another trivial solution of the Burgers equation with linear damping by using the direct method of Clarkson and Kruskal [3].

In the actual physical situations, the inhomogeneity of the medium, the roughness or non smoothness of the fluid bottom, and the non-uniformity of the boundary must also be considered. The variable coefficient partial differential equations often provide more powerful and realistic model than their constant coefficient counterparts in several physical situations. It seems more meaningful to consider the variable coefficient Burgers equation with damping.

In 1991, Oliveri [14] considered a generalized Burgers' equation containing an arbitrary function of time

$$u_t + uu_x - u_{xx} + f(t)u = 0. \tag{1.2}$$

He proved that equation (1.2) possesses the Painlevé property if and only if $f(t) = 0$, i.e. it reduces to the classical Burgers equation. He also determined some classes of functional forms for the function $f(t)$ compatible with the existence of similarity solutions of equation (1.2) by means of the Lie group techniques.

Qu Changzheng [17] derived a further generalized Eq.(1.1) with variable nonlinear and dissipative coefficients, i.e.

$$u_t + b(t)uu_x + a(t)u_{xx} = 0. \tag{1.3}$$

which can provide more useful models in many complicated physical situations, such as the propagation of a long shock wave in an inhomogeneous two-layer shallow liquid [8]. The allowed transformations, symmetry classes, Painlevé property, and Bäcklund transformation have been discussed in [8, 17] by the application of the truncated Painlevé expansion and symbolic computation method. Hong found kink-type solitonic solution under the conditions $a(t), b(t) \sim e^{-\alpha_1 t}$ with $\alpha_1 \ll 1$. In the past two decades, many authors have studied the exact linearization, Bäcklund transformation, and similarity reduction of Burgers equation with variable coefficients by using various methods. For more information, we suggest readers to read the literature [2, 6, 13, 16, 21, 22] and references therein.

Wang, Zhang, Li et.al [23] considered the following generalized variable coefficient Burgers equation with linear damping term

$$u_t + \alpha(t)uu_x - \beta(t)u_{xx} + \gamma(t)u = 0, \tag{1.4}$$

which can describe the propagation of nonlinear waves in a liquid subject not only to thermal conductivity but also to convective diffusion effects associated with the