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A Compact Difference Scheme for Time-Space Fractional Nonlinear Diffusion-Wave Equations with Initial Singularity

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Abstract. In this paper, we present a linearized compact difference scheme for one-dimensional time-space fractional nonlinear diffusion-wave equations with initial boundary value conditions. The initial singularity of the solution is considered, which often generates a singular source and increases the difficulty of numerically solving the equation. The Crank-Nicolson technique, combined with the midpoint formula and the second-order convolution quadrature formula, is used for the time discretization. To increase the spatial accuracy, a fourth-order compact difference approximation, which is constructed by two compact difference operators, is adopted for spatial discretization. Then, the unconditional stability and convergence of the proposed scheme are strictly established with superlinear convergence accuracy in time and fourth-order accuracy in space. Finally, numerical experiments are given to support our theoretical results.

AMS subject classifications: 65M06, 65M12

Key words: Fractional nonlinear diffusion-wave equations, finite difference method, fourth-order compact operator, stability, convergence.

1 Introduction

In this paper, the following time-space fractional nonlinear diffusion-wave equation with initial boundary value conditions will be considered

$$ {}_0^C D_t^{\alpha} u(x,t) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^{\beta}}{\partial |x|^{\beta}} \right) u(x,t) + g(u) + f(x,t), \tag{1.1a} $$

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$$u(x,0)=0, \quad u_t(x,0)=0, \quad 0<x<L, \tag{1.1b}$$

$$u(0,t)=u(L,t)=0, \qquad 0<t\leq T, \tag{1.1c}$$

where $1<\alpha,\beta\leq 2$, $g(u)$ is a nonlinear function of u that fulfills the Lipschitz condition with $g(0)=0$, $f(x,t)$ is a known function, and ${}_0^C D_t^\alpha u(x,t)$ is the temporal Caputo fractional derivative of order α defined as

$${}_0^C D_t^\alpha u(x,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{\partial^2 u(x,s)}{\partial s^2}ds.$$

And $\frac{\partial^\beta u(x,t)}{\partial|x|^\beta}$ is the Riesz fractional derivative of order β defined as

$$\frac{\partial^\beta u(x,t)}{\partial|x|^\beta}=-\frac{1}{2\cos(\frac{\pi\beta}{2})}\Big({}_0^{RL}D_x^\beta u(x,t)+{}_x^{RL}D_L^\beta u(x,t)\Big),$$

where ${}_0^{RL}D_x^\beta u(x,t)$ and ${}_x^{RL}D_L^\beta u(x,t)$ are the left and right Riemann-Liouville fractional derivatives of order β defined as

$${}_0^{RL}D_x^\beta u(x,t)=\frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_0^x (x-z)^{1-\beta}u(z,t)dz$$

and

$${}_x^{RL}D_L^\beta u(x,t)=\frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_x^L (z-x)^{1-\beta}u(z,t)dz,$$

respectively.

Remark 1.1. In the case of nonhomogeneous initial conditions, such as $u(x,0)=\varphi(x)\neq 0$ and $u_t(x,0)=\psi(x)\neq 0$. To homogenize the initial value conditions, we can use the following transformation

$$\hat{u}(x,t)=u(x,t)-\varphi(x)-t\psi(x).$$

Clearly, the nonhomogeneous boundary conditions can be similarly homogenized.

The time-space fractional diffusion-wave equation (1.1) can be considered as intermediate between parabolic diffusion equations and hyperbolic wave equations. It has been widely applied in the modeling of oxygen delivery through capillaries and anomalous relaxation in magnetic resonance imaging signal magnitude [1–3]. However, using currently available analytical methods, it is impossible to find an exact solution to Eq. (1.1) [4–6]. As a result, if Eq. (1.1) is to be used in practical modeling, effective numerical methods for solving it in the corresponding numerical simulations must be developed (see [7–11] for examples).