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Extended Milstein Approximation to the Stochastic Allen-Cahn Equation with Random Diffusion Coefficient Field and Multiplicative Noise

Xiao Qi[1,2,*]

[1] *School of Mathematical Sciences, Xiamen University, Xiamen 361005, China.*
[2] *Fujian Provincial Key Laboratory of Mathematical Modeling and High Performance Scientific Computing, Xiamen University, Xiamen 361005, China.*

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Abstract. This paper studies the stochastic Allen-Cahn equation driven by a random diffusion coefficient field and multiplicative force noise. A new time-stepping scheme based on a stabilized approach and Milstein scheme is proposed and analyzed. The proposed method is unconditionally stable in the sense that a discrete energy is dissipative when the multiplicative noise is absent. The strong convergence rate of a spatio-temporal fully discrete scheme is derived. Numerical experiments are finally reported to confirm the theoretical result and show that the new scheme is much more robust than the classical semi-implicit Euler-Maruyama scheme, especially when the interface width parameter is small.

AMS subject classifications: 60H15, 60H35, 65C50

Key words: Stochastic Allen-Cahn equation, multiplicative noise, strong convergence, extended Milstein scheme, stability.

1 Introduction

The Allen-Cahn equation was originally introduced by Allen and Cahn in [1] as a mathematical model to describe the motion of anti-phase boundaries in crystalline solids, and has been widely used as a fundamental equation in many complicated moving interface problems in materials science and fluid dynamics, see, e.g., [2, 17, 18, 27] and references therein. However, due to the existence of uncertainty stemming from various sources such as thermal fluctuation, impurities of materials and so on, it is often necessary to consider stochastic effects and to study the impact of noise on the phase transition process. As an active area of research, the numerical study of the stochastic Allen-Cahn equation has attracted increasing attention over the past decades, see, e.g., [7, 9, 10, 19, 32, 33, 42, 43, 48] and references therein. Although much progress has been made in the numerical approximation of the stochastic

*Corresponding author. *Email address:* `qixiao@stu.xmu.edu.cn` (Qi X)

partial differential equations (SPDEs) with globally Lipschitz nonlinearity, see, e.g., monographs [36, 41, 51] and references therein, the numerical analysis and simulation of the SPDEs with non-globally Lipschitz nonlinearity are still far from being satisfactory due to the essential difficulties caused by local Lipschitz nonlinearity, infinite dimensional operator, and driving noise, see, e.g., [6, 8, 13, 14, 21, 23–26, 28, 39, 40] and references therein. The current work focuses on the numerical investigation of stochastic Allen-Cahn equation perturbed by a smooth random diffusion coefficient field as well as multiplicative force noise, and aims to propose and analyze an efficient numerical method for this equation.

The stochastic Allen-Cahn equations are commonly classified into two categories in terms of the different force noises, namely those with additive noise or multiplicative noise. We mention here some recent work on the numerical approximation of the stochastic Allen-Cahn equations with additive noise. For example, Qi et al. [43] proposed a backward Euler-Galerkin finite element full discretization scheme for the stochastic Allen-Cahn equation perturbed by additive Gaussian noise, and showed the optimal spatio-temporal strong convergence rate. Bréhier et al. [9] analyzed an explicit temporal splitting scheme for the stochastic Allen-Cahn equation driven by additive noise, and obtained strong convergence rates of order $1/4$. Bréhier and Goudenège [10] studied the temporal discretization of the equation with space-time white noise, and proved that the weak convergence rate is $1/2$. Some other related works include Kovács et al. [33], Becker and Jentzen [8], Cui et al. [13], Wang [48], and Wang et al. [11], in which different schemes were constructed and analyzed for different types of additive noises. The case of multiplicative noise is generally more subtle and challenging, and has received extensive attention in recent years. For instance, Feng et al. [19] investigated a finite element approximation to the stochastic Allen-Cahn equation with gradient-type multiplicative noise that is white in time and correlated in space. Majee et al. [42] deduced an optimal strong error estimate for a modified implicit Euler scheme. Besides the form of Allen-Cahn potential functional, there has also been a paper on the numerical analysis of more general SPDEs with non-globally Lipschitz nonlinear terms. In this regard, we mention the work, for example, by Huang et al. [23] on a spatio-temporal fully discrete scheme for approximating a class of SPDEs with local Lipschitz coefficients, and the work by Liu et al. [40] on a general theory of optimal strong error estimation for SPDEs with monotone drift driven by a multiplicative infinite-dimensional Wiener process.

In this paper, we consider the stochastic Allen-Cahn equation with both multiplicative noise and random diffusion coefficient field. The purpose is to propose and analyze an effective numerical scheme for this equation. The idea is to make a combination of the stabilized approach and Milstein scheme, where the former has been found useful in constructing the stable scheme for deterministic Allen-Cahn equation [45], and the latter has been applied to numerically solve SPDEs satisfying a certain commutativity type condition [29, 34, 40]. To the best of our knowledge, this new method has not yet been considered and analyzed in the literature for the underlying problem. The main contributions/novelties of this paper are summarized as follows:

1) The well-posedness of the underlying equation is established. That is, the existence, uniqueness, and the stability of the mild solution is proved.

2) The proposed time-stepping method is computationally very efficient since only one second-order linear equation with random variable coefficients needs to be solved at each time