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The Exact Limits and Improved Decay Estimates for All Order Derivatives of Global Weak Solutions of Three Incompressible Fluid Dynamics Equations

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Abstract First of all, the author accomplishes the exact limits for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations, the n-dimensional incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation. Secondly, by making use of the exact limits, he establishes the improved decay estimates with sharp rates for all order derivatives of the global weak solutions, for all sufficiently large t. The author proves these results by making use of existing ideas, existing results and several new, novel ideas.

Keywords Incompressible fluid dynamics equations, global weak solutions, all order derivatives, exact limits, improved decay estimates with sharp rates

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1. Introduction

Consider the n-dimensional incompressible magnetohydrodynamics equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{u}-\frac{1}{\mathrm{RE}}\triangle\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{A}\cdot\nabla)\mathbf{A}+\nabla P=\mathbf{f}(\mathbf{x},t),\\
&\frac{\partial}{\partial t}\mathbf{A}-\frac{1}{\mathrm{RM}}\triangle\mathbf{A}+(\mathbf{u}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{u}=\mathbf{g}(\mathbf{x},t),\\
&\nabla\cdot\mathbf{u}=0,\qquad \nabla\cdot\mathbf{f}=0,\qquad \nabla\cdot\mathbf{A}=0,\qquad \nabla\cdot\mathbf{g}=0,
\end{aligned}$$

the n-dimensional incompressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u}-\alpha\triangle\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}(\mathbf{x},t),\qquad \nabla\cdot\mathbf{u}=0,\qquad \nabla\cdot\mathbf{f}=0,$$

and the two-dimensional incompressible dissipative quasi-geostrophic equation

$$\frac{\partial}{\partial t}u+\alpha(-\triangle)^{\rho}u+J(u,(-\triangle)^{-1/2}u)=f(\mathbf{x},t).$$

These incompressible fluid dynamics equations play very important roles in applied mathematics. See Caffarelli, Kohn, Nirenberg [5], Leray [25], Temam [40] and [41] for the physical backgrounds. It is well known that there exists a unique

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global smooth solution or a global weak solution to each equation, under certain conditions. Even if the dimension is high, the nonlinear couplings are strong and the initial functions and the external forces are large, the global weak solutions exist. Moreover, there hold some elementary uniform energy estimates for the global weak solutions. The global weak solutions become small enough and sufficiently smooth after a long time T.

The elementary decay estimates with sharp rates have been established very well for the global weak solutions of these equations. For very similar n-dimensional incompressible fluid dynamics equations, the existence of a global smooth solution or the existence of a global weak solution, the elementary uniform energy estimates and the elementary decay estimates with sharp rates have also been established. For more results on the incompressible magnetohydrodynamics equations, the incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation, please see all the references [1]-[44] for these known related results.

Let n be a positive integer, such that $2 \le n \le 5$. Let $\alpha > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, $0 < \delta < 4$, $0 < \varepsilon < 1$, $0 < \rho < 1$ and $T > 0$ be positive constants. Let $m \ge 0$ be a constant.

We will accomplish the exact limits for all order derivatives of the global weak solutions of these incompressible fluid dynamics equations. We may use the global smooth solution of the corresponding linear problem to approximate the global weak solutions of the nonlinear problem. We also establish the improved decay estimates with sharp rates for all order derivatives of the global weak solutions.

The exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions are very challenging problems, because if the spatial dimension is high and the nonlinear functions are strong, the existence and uniqueness of the global smooth solution may be unknown and there exist no available uniform energy estimates for any order derivatives of the global weak solutions.

The main purposes of the next three sections are to accomplish the exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions, which are also local smooth solutions on some unbounded interval (T, ∞), for all of the incompressible fluid dynamics equations mentioned above. In each section, we will make mathematical assumptions, provide the precise statements of the main results. All the results on the global weak solutions of these incompressible fluid dynamics equations are completely new.

The main ideas and the main steps in the proofs of the main results for the n-dimensional incompressible magnetohydrodynamics equations, the n-dimensional incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation are almost the same, although some details may be slightly different. To keep the paper from being too long, we will only provide a sketch of the proofs of the main results for the n-dimensional incompressible magnetohydrodynamics equations. We will skip the proofs of the main results for the n-dimensional incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation.

Definition 1.1. Let $\phi \in L^1(\mathbb{R}^n)$. Define the Fourier transformation

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} \exp(-\mathrm{i}\mathbf{x} \cdot \xi)\phi(\mathbf{x})\mathrm{d}\mathbf{x}, \qquad \xi \in \mathbb{R}^n.$$