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Uniqueness of Limit Cycles in a Predator-Prey Model with Sigmoid Functional Response*

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Abstract In this paper, we prove that a predator-prey model with sigmoid functional response and logistic growth for the prey has a unique stable limit cycle, if the equilibrium point is locally unstable. This extends the results of the literature where it was proved that the equilibrium point is globally asymptotically stable, if it is locally stable. For the proof, we use a combination of three versions of Zhang Zhifen's uniqueness theorem for limit cycles in Liénard systems to cover all possible limit cycle configurations. This technique can be applied to a wide range of differential equations where at most one limit cycle occurs.

Keywords Limit cycle, predator-prey system, Liénard equation, Sigmoid functional response

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1. Introduction

In [15] a predator-prey system was discussed with a sigmoid functional response $p(x)$

$$\begin{aligned} \frac{dx(t)}{dt} &= h(x) - p(x)y, \\ \frac{dy(t)}{dt} &= -\mu y + \beta p(x)y \end{aligned} \tag{1.1}$$

with $h(x) = rx(1 - \frac{x}{K})$ and $p(x) = \frac{x^2}{a+bx+cx^2}$.

The parameters are real and positive. The growth of the prey population is chosen to be logistic with r, the intrinsic growth rate and K, the carrying capacity. The parameter μ is the predator death rate, and β is the constant conversion rate of the predator after eating the prey. The parameters a, b and c in the functional response do not have an obvious biological meaning and are considered to be fit

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through a phenomenological approach based on observed data, a method followed by Holling in [8].

This type of functional response for a predator-prey model was first introduced by Holling in his ground-breaking paper [8]. The typical approach in predator-prey systems is to take a functional response which is monotonically increasing as a function of the prey density, bounded and without inflection points, e.g., the Holling II functional response function.

However, recently, more research has been done on the case where the functional response has a sigmoid shape with one inflection point. The functional response in this paper can be seen as a generalization of the Holling III functional response. Other variations of this response function similar to the one in this paper can be found in [1–7, 9, 10, 13].

It is assumed that system (1.1) has a positive equilibrium point A with $x^* > 0$ and $y^* > 0$ such that $h(x^*) - p(x^*)y^* = 0$, $-\mu + \beta p(x^*) = 0$ (here, positive refers to the fact that the prey and predator density have a positive value). It is not difficult to see that A is unique when it exists. In the following, we will always assume that the parameters of the system are such that there is exactly one positive equilibrium point A, because the dynamics of the system become trivial without it.

The main result of [15] was the proof that the system is globally asymptotically stable, when A is locally stable. Effectively, this follows from the fact that the system does not have limit cycles, is bounded and does not have attracting equilibrium points on the coordinate axes in the phase plane.

The aim of this paper is to show that system (1.1) has exactly one limit cycle, if A is locally unstable. In the process, we will give a simplified proof of the result in [15] for the non-existence of the limit cycle, when A is locally stable. The methods we use are a combination of three flavours of Zhang Zhifen's theorem. The reason three different versions are needed is that the divergence of the associated Liénard system is zero on two vertical lines in the phase plane. Since it is not known a priori which of these lines will be crossed by a limit cycle, different configurations need to be considered. In each case, a different version of Zhang Zhifen's theorem is needed.

From a mathematical point of view, the choice of a sigmoidal functional response function is related to a question discussed in a few papers [11, 12, 16], where the influence of the convexity of the functional response function on the number of limit cycles was explored. For example, it was shown in [16] that a functional response function with a change in convexity, similar to a sigmoidal function, leads to more limit cycles than the case where it is just concave down. On the other hand, in [12], the cases were observed with more than one limit cycle in the concave down case. This paper shows that even in a class with a sigmoidal functional response function that is more general than the traditional Holling III function, not more than one limit cycle will occur. This suggests that it is not necessarily the convexity of the function that is driving the number of limit cycles in a Gause predator-prey system.

In Section 2, we introduce the theorems for limit cycles in Liénard systems which we will use. Section 3 contains the relation between system (1.1) and Liénard systems. In Section 4, we show the non-existence of limit cycles in the case of the stable equilibrium point A. Then, the main result will be given in Section 5, where we show the uniqueness of limit cycles when A is unstable.