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Existence Results for the Higher-Order Weighted Caputo-Fabrizio Fractional Derivative*

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Abstract By the definition of the higher-order fractional derivative, we explore the central properties of the higher-order Caputo-Fabrizio fractional derivative and integral with a weighted term. Furthermore, by dint of Schaefer's fixed point theorem, α-ψ-Contraction theorem, etc., we establish the existence of solutions for nonlinear equations. We also give three examples to make our main conclusion clear.

Keywords Higher-order weighted fractional derivative, Caputo-Fabrizio derivative, existence

MSC(2010) 26A33, 35A01, 47H10.

1. Introduction

During the past decades, the Caputo fractional derivative (CFD) has been investigated by many scholars (see [13,18]). In the last few years, a large number of essays about a novel fractional derivative, Caputo-Fabrizio fractional derivative (CFFD), have emerged, and this kind of derivative has a better nature than the usual fractional derivative (see [1–3,5,7,9,12,15,16,19]). For instance, in 2020, Eiman et al., dealt with the nether class of fractional differential equations involving the CFFD and obtained the existence theory

$$\begin{cases} {}^{CF}_{0}D^{\theta}_{x}u(x) = f(x, u(x), {}^{CF}_{0}D^{\theta}_{x}u(x)), \quad x \in [0, T] = \mathbb{J}, \\ u(0) = u_0, \quad u_0 \in \mathbb{R}, \end{cases}$$

where $\theta \in (0, 1]$, f: $\mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (see [9]). In 2021, Abbas et al., investigated the existence of solutions for the following Cauchy problem of Caputo-Fabrzio impulsive fractional differential equations

$$\begin{cases} ({}^{CF}D^{r}_{t_k}u)(t) = f(t, u(t)); t \in I_k, k = 0, \cdots, m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); k = 1, \cdots, m, \\ u(0) = u_0, \end{cases}$$

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where $I_0 = [0, t_1]$, $I_k = (t_k, t_{k+1}]$, $k = 1, \cdots, m$; $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $u_0 \in \mathbb{R}$, and $f : I_k \times \mathbb{R} \to \mathbb{R}$, $k = 0, \cdots, m$, $L_k : \mathbb{R} \to \mathbb{R}$; $k = 1, \cdots, m$ are given continuous functions, ${}^{CF}D^r_{t_k}$ is the Caputo-Fabrizio fractional derivative of order $r \in (0, 1)$ (see [2]). In 2022, Abbas et al., investigated the existence of solutions for the Cauchy problem of Caputo-Fabrzio fractional differential equations without instantaneous impulses

$$\begin{cases} ({}^{CF}D^r_{s_k}u)(t) = f(t, u(t)); t \in I_k, k = 0, \cdots, m, \\ u(t) = g_k(t, u(t_k^-)); \text{if } t \in J_k, k = 1, \cdots, m, \\ u(0) = u_0 \in \mathbb{R}, \end{cases}$$

where $I_0 := [0, t_1]$, $J_k := (t_k, s_k]$, $I_k := (s_k, t_{k+1}]$; $k = 1, \cdots, m$, and $f : I_k \times \mathbb{R} \to \mathbb{R}$, $g_k : J_k \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $0 = s_0 < t_1 \le s_1 < t_2 \le s_2 < \cdots \le s_{m-1} < t_m \le s_m < t_{m+1} = T$ (see [3]).

Abreast of the times, in 2022, Fernandez et al., conducted a formal study of weighted fractional calculus, and emphasized the importance of the conjugation relationships with the classical Riemann-Liouville fractional calculus (see [11]). For the study of Caputo-Fabrizio fractional derivative (CFFD) in the weighted field, in 2019, Al-Refai and Jarrah first proposed the weighted Caputo-Fabrizio fractional derivative (WCFFD) of order 0 to 1, and demonstrated the existence and uniqueness of the nonlinear fractional initial value problem

$$\begin{cases} (D^\alpha_{a,[z,w]}f)(t) = g(t, f), t > a, 0 < \alpha < 1, \\ f(a) = f_0 \in \mathbb{R}, \end{cases}$$

where $D^\alpha_{a,[z,w]}$ is the WCFFD (see [4]). In 2020, Wu, Chen and Deng studied the existence and stability of solutions for the WCFFD type differential equations of order 0 to 1 (see [20]). However, fewer papers are on the higher-order WCFFD.

In this paper, we are concerned with the existence of solutions for the following nonlinear equations

$$\begin{cases} (D^r_{a,[z,w]}y)(t) = \xi(t, y(t)), \\ y^{(k)}(a) = 0, \quad k = 0, 1, 2, \cdots, n-1, \\ y^{(n)}(a) = 1, \end{cases} \tag{1.1}$$

where $1 \le n < r < n+1$, $D^r_{a,[z,w]}$ is the higher order WCFFD, and $y \in AC^n([a, T], \mathbb{R})$, ξ are binary continuous functions

$$\begin{cases} D^r_{a,[z,w]}(y(t) - \varpi(t, y(t)) = \xi(t, y(t)), \\ (y - \varpi)^{(k)}(a) = 0, \quad k = 0, 1, 2, \cdots, n-1, \\ (y - \varpi)^{(n)}(T) = 0, \end{cases} \tag{1.2}$$

where $y - \varpi \in AC^n([a, T], \mathbb{R})$, and ϖ are binary continuous functions

$$\begin{cases} D^r_{a,[z,w]}\frac{y(t)}{\varphi(t, y(t))} = \xi(t, y(t)), \\ y^{(k)}(a) = 0, \quad k = 0, 1, 2, \cdots, n-1, \\ y^{(n)}(a) = 1, \end{cases} \tag{1.3}$$

where $\frac{y}{\varphi} \in AC^n([a, T], \mathbb{R})$, and φ are binary continuous functions. Here, $AC([a, T], \mathbb{R})$ is Banach space, which contains all absolutely continuous functions from $[a, T]$ into $\mathbb{R}$, provided with the usual maximum norm. $AC^n([a, T], \mathbb{R}) = \{x : [a, T] \to$