Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 5, Number 4, December 2023, 753–762 DOI:10.12150/jnma.2023.753

On Some SEIRS Epidemic Models*

Valery G. Romanovski[1,2,3,†]

Abstract We discuss a few variations of the SEIRS epidemic model. How basic dynamical properties of the models can be derived by using some tools of the computer algebra system Mathematica is shown, and how invariant surfaces of the system can be found by using computer algebra system Singular is explained. Some numerical simulations are presented as well.

Keywords Epidemiology, SEIRS model, stability of solution, invariant surface

MSC(2010) 92D30, 34D20.

1 Introduction

The first applications of mathematical methods to the analysis of epidemics are associated with the works of D. Bernoulli, I. Lambert and P. S. Laplace. It appears that modern mathematical models of epidemiology go back to the work of R. Ross, published in 1911, on the study of the spread of malaria [12], and to the SIR model proposed in 1927 by W. Kermack and A. McKendrick [10]. The SIR model is based on the division of the entire population into three groups of susceptible, infected and recovered individuals, and describes the transition of individuals from the group of susceptible to the group of infected and then recovered. Mathematically, it is given as a system of differential equations that describe the change in the size of each of these population groups over time. However, the SIR model does not take into account the presence of the incubation period of the disease, i.e., it is assumed that a person who has had contact with a sick person immediately falls ill. This shortcoming is overcome in the SEIR model, which incorporates a group of contacts (exposed) (see, for example, [8, 9]). Thus, in the process of infection, a person susceptible to the disease first becomes exposed, and only after some time becomes infected.

A further development of the model is the SEIRS (Susceptible–Exposed–Infectious–Recovered–Susceptible) model (see e.g., [1]). The model considers the population divided in four groups: susceptible (S), exposed (E), infectious (I) and recovered

[†]The corresponding author.
Email: valerij.romanovskij@um.si (V. G. Romanovski)
[1]Center for Applied Mathematics and Theoretical Physics, University of Maribor, Mladinska 3, SI-2000 Maribor, Slovenia
[2]Faculty of Electrical Engineering and Computer Science, University of Maribor, Koroška c. 46, 2000 Maribor, Slovenia
[3]Faculty of Natural Science and Mathematics, University of Maribor, Koroška c. 160, 2000 Maribor, Slovenia
*The author was supported by Slovenian Research Agency (Program P1-0306) and "Development of an intelligent system for assessing the development of COVID-19 epidemics and other infections in Kazakhstan" of al-Farabi Kazakh National University (Grant No. AP09260317).

(R). However, it is assumed that recovered people may become susceptible again. Mathematically, the simplest SEIRS model is described by the following system of differential equations:

$$\begin{aligned}\dot{S} =& -\beta SI/N + \omega R,\\ \dot{E} =& \beta SI/N - \sigma E,\\ \dot{I} =& \sigma E - \gamma I,\\ \dot{R} =& \gamma I - \omega R.\end{aligned} \tag{1.1}$$

In the model, the infectious rate β is the rate of spread which represents the probability of transmitting the disease between a susceptible and an infectious individual. The incubation rate σ is the rate of latent individuals becoming infectious. Recovery rate $\gamma = 1/d$ is determined by the average duration d of the infection, and ω is the rate the recovered individuals return to the susceptible state due to a loss of immunity. $N = S + I + E + R$ is the total population and since in model (1.1), it is assumed that the population is closed with no births and deaths, and N is a constant.

Some recent extensions of the model taking into account vaccination or time-delay are presented (e.g., in [2,6,13]), and works are referenced there.

In this paper, we review some main dynamical properties of a few variations of the model, and show how they can be easily derived by using certain tools of the computer algebra system MATHEMATICA. We also clarify the behavior of the model with the vital dynamics comparing our computational results with the ones obtained in [1].

2 Singular points and invariant surfaces in the SEIRS model

Introducing the notation

$$S = x_1,\ E = x_2,\ I = x_3,\ R = x_4, \tag{2.1}$$

we write system (1.1) as

$$\begin{aligned}\dot{x}_1 =& -(\beta x_1 x_3)/N + x_4\omega,\\ \dot{x}_2 =& -\sigma x_2 + (\beta x_1 x_3)/N,\\ \dot{x}_3 =& \sigma x_2 - \gamma x_3,\\ \dot{x}_4 =& \gamma x_3 - x_4\omega.\end{aligned} \tag{2.2}$$

System (2.2) has a line filled with steady states

$$x_1 = \frac{\gamma N}{\beta},\quad x_2 = \frac{\gamma x_3}{\sigma},\quad x_4 = \frac{\gamma x_3}{\omega}$$

and the first integral

$$\Psi = x_1 + x_2 + x_3 + x_4. \tag{2.3}$$

After rescaling of the phase variables

$$x_1 = \frac{x_1}{N},\quad x_2 = \frac{x_2}{N},\quad x_3 = \frac{x_3}{N},\quad x_4 = \frac{x_4}{N},$$