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Positive Solutions for Hilfer Fractional Differential Equation Boundary Value Problems at Resonance*

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Abstract In this paper, we investigate the positive solutions for Hilfer fractional differential equation boundary value problems at resonance. First, we give the expression of the solution with Mittag-Leffler function. Next, we obtain the existence of the positive solutions by using fixed point index theorem. Finally, we give relevant examples to prove our main results.

Keywords Fractional differential equation, resonance, Hilfer fractional derivative, positive solution, fixed point index

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1. Introduction

In this paper, we consider the following Hilfer fractional differential equation boundary value problem (FBVP) at resonance

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t) + f(t,x(t)) = 0, \quad t\in(0,1), \\ x(0)=0, x(1)=\eta x(\xi), \end{cases} \tag{1.1}$$

where $1<\alpha<2$, $0\le\beta\le1$, $1<\gamma=\alpha+\beta(2-\alpha)<2$, $0<\xi<1$, and $\eta\xi^{\gamma-1}=1$. $D_{0^+}^{\alpha,\beta}$ is Hilfer fractional derivative and $f:[0,1]\times[0,+\infty)\to\mathbb{R}$ is continuous.

Boundary value problem (BVP) (1.1) is resonant as the corresponding homogeneous BVP

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t) = 0, \quad t\in(0,1), \\ x(0)=0, x(1)=\eta x(\xi) \end{cases} \tag{1.2}$$

has a nontrivial solution $ct^{\gamma-1}$, where $c\in\mathbb{R}, c\neq 0$.

Fractional differential equations can describe the objective world more accurately than integer differential equations, so they are widely used in physical mechanics, biomedicine, viscoelastic system, finance and other aspects (see [1,9,22]). There are various definitions for fractional integrals and derivatives, and the most commonly used are Riemann-Liouville derivative and Caputo derivative (see [5,8,10,14,19]).

Hilfer fractional derivative is an interpolation between Riemann-Liouville derivative and Caputo fractional derivative. When $\beta=0$, it corresponds to Riemann-Liuoville fractional derivative. When $\beta=1$, it corresponds to Caputo fractional

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derivative. Therefore, it is important to study Hilfer fractional derivative, which includes both Riemann-Liuoville and Caputo fractional derivative. Recently, many scholars have devoted themselves to the study on fractional differential equations with Hilfer derivative and have obtained abundant results (see [6,11–13,16,20]).

For the special case of Hilfer fractional derivative $\beta = 0$ and $\beta = 1$, Wang studied BVP (1.1) in the sense of Riemann-Liouville fractional derivative (see [18]), and Yang studied BVP (1.1) in the sense of Caputo fractional derivative (see [21]).

Wang investigated the following Hilfer FBVP at non-resonance by fixed point theorem (see [16])

$$\begin{cases} D_{a^+}^{\alpha,\beta} u(t) = f(t, u(t)), \quad t \in (a, b], \\ I_{a^+}^{1-\gamma} u(a^+) = \sum\limits_{i=1}^{m} \lambda_i u(\tau_i), \quad \tau_i \in (a, b], \end{cases}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\Gamma(\gamma) \neq \sum\limits_{i=1}^{m} \lambda_i (\tau_i - a)^{\gamma-1}$, and $D_{a^+}^{\alpha,\beta}$ is Hilfer fractional derivative.

Yong investigated the following Hilfer FBVP at resonance by upper and lower solutions (see [12])

$$\begin{cases} D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t)), \quad t \in (0, T] = J \setminus \{0\}, \\ I_{0^+}^{1-\gamma} x(0) = \sum\limits_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in J, \end{cases}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\Gamma(\gamma) = \sum\limits_{i=1}^{m} c_i (\tau_i)^{\gamma-1}$, and $D_{0^+}^{\alpha,\beta}$ is Hilfer fractional derivative.

First, most of studies on the Hilfer fractional derivative are considered in the case of $0 < \alpha < 1$ (see [6,11–13,16,20]), while in this paper, we consider the case of the higher order $1 < \alpha < 2$. Higher order differential equations can be applied to the establishment of control systems and diffusion systems. For example, Nigmatullin derived fractional diffusion-wave equation (see [17])

$${}_0D_t^{\alpha} u(x, t) = \frac{d^2 u(x, t)}{dx^2}.$$

When $\alpha = 1$, it is the traditional diffusion equation, when $\alpha = 2$, it is the traditional wave equation, and when $1 < \alpha < 2$, it is an intermediate state between diffusion and wave.

Secondly, the studies on the Hilfer FBVPs are limited to the non-resonant cases (see [11,13,16,20]), and there is little research on the resonant case (see [6,12]). Especially, as far as we know, no relevant results have been obtained for the existence of positive solutions so far. The study of positive solutions has important theoretical significance and practical value in practical problems. For example, the optimal control problem the HIV model can be abstracted into the following fractional differential equations of the same order (see [15])

$$\begin{cases} {}_0^C D_T^{\alpha} x(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \end{cases}$$

where $0 < \alpha < 1$, $x(t)$ is the n-dimensional state vector, $u(t)$ is the m-control vector, and f is the n-dimensional vector function. For a given control function,