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Hopf Cyclicity of a Class of Liénard-Type Systems*

Hongwei Shi[1] and Changjian Liu[1,†]

Abstract The Hopf cyclicities of some smooth polynomial, rational polynomial and piecewise smooth Liénard systems are studied. For two Liénard systems with the same damping term and different restoring (or potential) terms, we provide sufficient conditions that the two systems have the same Hopf cyclicity. Then, some examples are given to illustrate the efficiency and applicability of our results.

Keywords Hopf cyclicity, polynomial and rational polynomial Liénard systems, smooth and piecewise smooth systems.

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1. Introduction

Consider the following Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.1}$$

which appears in many models of classical Newtonian mechanics. Usually, we call $f(x)$ a damping term and $g(x)$ a restoring term (or potential term). Historically, Liénard equation is not only closely related to a large number of practical applications (see e.g., [8, 20]), but also plays an important role in the theoretical studies of qualitative theory (see e.g., [1, 6, 10, 14, 16, 19, 21, 23, 26–29]).

The above equation (1.1) is equivalent to the following planar differential system (called Liénard system)

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{1.2}$$

where $F(x) = \int_0^x f(x)dx$. Here, we assume that $g(0) = 0$, $g'(0) > 0$, so that the origin is a center or focus.

In the qualitative theory of differential systems, an important open problem is to determine the maximum number of limit cycles bifurcating from a center or a focus, which is related to the local version of Hilbert's 16th problem. One way to study the limit cycles of the system (1.2) is the Hopf bifurcation, that is, to study the small-amplitude limit cycles. Usually, the maximum number of small limit cycles obtained by Hopf bifurcation is called Hopf cyclicity. When $f(x)$ and $g(x)$ are polynomials of degree n and m respectively, we denote the Hopf cyclicity

[†]The corresponding author.
Email address: liuchangj@mail.sysu.edu.cn (C. Liu), shihw7@mail2.sysu.edu.cn (H. Shi)
[1]School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai, Guangdong 519082, China
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of system (1.1) by $H_L^s(m,n)$. Many results on $H_L^s(m,n)$ are obtained by computing Lyapunov coefficients. However, due to the difficulty of the problem, most of the results are obtained under the strict assumptions of $F(x)$ or $g(x)$, for example, the degrees and coefficients in $F(x)$ or $G(x)$ are fixed, etc. The results of $H_L^s(m,n)$ are rarely reported for arbitrary values of m or n.

For $m=2$, $g(x)$ is a quadratic polynomial. Without loss of generality, we can assume system (1.2) has the form

$$\dot{x} = y - F(x), \quad \dot{y} = -x(x+1), \tag{1.3}$$

where

$$F(x) = F_1(x) = \sum_{i=1}^{n+1} a_i x^i.$$

Han [11, 12], Christopher and Lynch [4] independently investigated system (1.3), and obtained respectively that the Hopf cyclicity $\mathbb{H}_L^s(2,n) = \left[\frac{2n+1}{3}\right]$ at the origin for $n \geq 1$. Moreover, it was verified in [4] that $\mathbb{H}_L^s(2,n) = \mathbb{H}_L^s(n,2)$. When the damping term

$$F(x) = F_2(x) = \begin{cases} \sum_{i=1}^{n+1} a_i^+ x^i, & x > 0, \\ \sum_{i=1}^{l+1} a_i^- x^i, & x \leq 0 \end{cases}$$

in system (1.3) is a piecewise smooth polynomial in x of degree l and n, Tian and Han [24] obtained the cyclicity $\left[\frac{3l+2n+4}{3}\right]$ (resp., $\left[\frac{3n+2l+4}{3}\right]$) for $n \leq l$ (resp., $n \geq l$) of the origin.

In the case of $m=3$, according to the conditions $g(0)=0$ and $g'(0)>0$, let us assume that $g(x) = x + b_1 x^2 + b_2 x^3$. Christopher and Lynch [4] also obtained the following results, when $F(x) = F_1(x)$:

(i) if $b_1 = 0$, the cyclicity $\mathbb{H}_L^s(3,n) = \mathbb{H}_L^s(n,3) = \left[\frac{n}{2}\right], n \geq 1$;

(ii) if $b_1 \neq 0$, by scaling x and y simultaneously, $g(x) = x + x^2 + bx^3$, the cyclicity $\mathbb{H}_L^s(3,n) = \mathbb{H}_L^s(n,3) = 2\left[\frac{3(n+2)}{8}\right], 1 \leq n \leq 50$.

Tian, Han and Xu [25] studied the system with a special cubic restoring term

$$\dot{x} = y - F(x), \quad \dot{y} = -\frac{1}{2}x(x+1)(x+2). \tag{1.4}$$

It was proved that the Hopf cyclicity is $\mathbb{H}_L^s(3,n) = \left[\frac{3n+2}{4}\right]$ (resp., $\left[\frac{2l+n+2}{2}\right]$ as $l \geq n$ or $\left[\frac{2n+l+2}{2}\right]$ as $l \leq n$), if $F(x) = F_1(x)$ (resp., $F_2(x)$), $n \geq 1$.

Recently, Sun and Yu [22] have considered a Liénard system with a quintic restoring term, which is equivalent to the following form

$$\dot{x} = y - F(x), \quad \dot{y} = x\left(x + \frac{1}{2}\right)(x-1)^3, \tag{1.5}$$

where $F(x) = F_1(x)$ or $F_2(x)$. The Hopf cyclicity of system (1.5) at the origin is $\mathbb{H}_L^s(5,n) = \left[\frac{2n+1}{3}\right]$ (resp., $\left[\frac{3l+2n+4}{3}\right]$, if $n \leq l$ or $\left[\frac{3n+2l+4}{3}\right]$, if $n \geq l$) for $F(x) = F_1(x)$ (resp., $= F_2(x)$), $n \geq 1$.

Notice that systems (1.3) and (1.5) have the same Hopf cyclicity at the origin for $F(x) = F_1(x)$ or $F_2(x)$. Is this a coincidence or does it contain some structural rules?