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A DIRECT DISCONTINUOUS GALERKIN METHOD FOR TIME FRACTIONAL DIFFUSION EQUATIONS WITH FRACTIONAL DYNAMIC BOUNDARY CONDITIONS*

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Abstract

This paper deals with the numerical approximation for the time fractional diffusion problem with fractional dynamic boundary conditions. The well-posedness for the weak solutions is studied. A direct discontinuous Galerkin approach is used in spatial direction under the uniform meshes, together with a second-order Alikhanov scheme is utilized in temporal direction on the graded mesh, and then the fully discrete scheme is constructed. Furthermore, the stability and the error estimate for the full scheme are analyzed in detail. Numerical experiments are also given to illustrate the effectiveness of the proposed method.

Mathematics subject classification: 65M12.
Key words: Time fractional diffusion equation, Numerical stability, Convergence.

1. Introduction

Classical partial differential equations (PDE) with the dynamic boundary conditions (DBC) have been considered in [1, 2] by some physicists to simulate the interaction among fluids with the walls of domain. Recently, there are also some reports on their numerical solutions [3–5]. However, the research of efficient numerical methods for the time fractional partial differential equations (TFPDE) with the fractional dynamic boundary conditions (FDBC) has not been developed very much. To the best of our knowledge, the only work [6] on this topic presented a Rothe's approach for the source identification of the time fractional wave equations with FDBC. In this paper, we will study the following time fractional diffusion equations with FDBC:

$$
\begin{aligned}
&{}_0^C D_t^\alpha u(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) + f(\boldsymbol{x},t), && (\boldsymbol{x},t)\in\Omega\times(0,T], && (1.1a)\\
&u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), && \boldsymbol{x}\in\bar{\Omega}, && (1.1b)\\
&\mu\, {}_0^C D_t^\alpha u(\boldsymbol{x},t) = -\lambda u(\boldsymbol{x},t) + \varsigma\Delta_{\partial\Omega} u(\boldsymbol{x},t) - \partial_{\mathbf{n}} u(\boldsymbol{x},t) + f_{\partial\Omega}(\boldsymbol{x},t), && (\boldsymbol{x},t)\in\partial\Omega\times(0,T], && (1.1c)
\end{aligned}
$$

where $0<\alpha<1$, $\boldsymbol{x}:=(x,y)\in\Omega$ with Ω is a bounded domain in $\mathbb{R}^2$, $\bar{\Omega}$ is the closure of Ω, $\partial\Omega$ is the boundary of Ω, μ, λ, ς are non-negative constants, f_Ω and $f_{\partial\Omega}$ are given functions, and ${}_0^C D_t^\alpha$ denotes the left Caputo fractional derivative operator

$$
{}_0^C D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}s}u(s)\right)\mathrm{d}s.
$$

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Here, Δ is the classical Laplace operator along with $\Delta_{\partial\Omega}$ is Laplace-Beltrami operator [7], $\partial_{\mathbf{n}}u$ represents the exterior normal derivative of u on $\partial\Omega$.

Anomalous heat diffusion appears in many problems, and can be modeled by the fractional diffusion equation (1.1a). Here, u is the temperature, $\partial_{\mathbf{n}}u$ is used to represent the heat flux, f and $f_{\partial\Omega}$ represent the heat source in the domain and on the boundary respectively. It is worth noting that the anomalous rate of change for the heat with the time should be taken into account both in the region and on the boundary. The feature of anomalous diffusion is the nonlinear growth of the mean squared displacement with time $\langle \boldsymbol{x}^2(t)\rangle \sim t^{\alpha}$, which deviates from the well-known property $\langle \boldsymbol{x}^2(t)\rangle \sim t$ of the Brownian motion [8]. Therefore, we use the time fractional derivative operator in (1.1c). Because the Laplace-Beltrami operator contains the tangential derivative, the heat flow along the boundary is allowed. The effect of the boundary condition (1.1c) is to send a "heat wave" into the region and an infinitesimal layer near the boundary. In fact, FDBC with the Caputo derivative is an extension of typical DBC. Especially, when $\alpha = 1$, (1.1) reduces to the parabolic problems with DBC in [5], whose derivation and physical interpretation can be found in [9].

Fractional differential equations have recently drawn growing attention because the diverse utilizations, such as dynamics in self-similar structures, engineering science, biological systems and so on [10–12]. In particular, as a valuable tool to model complex systems for anomalous diffusion transport, TFPDE with initial-boundary value conditions have been investigated by many scholars [13, 14]. Recently, the construction and analysis of discretization approach for TFPDE has obtained a series of results in numerical methods, see [15–18].

Discontinuous Galerkin (DG) method has been widely applied for solving PDE numerically owing to its flexibility and higher accuracy. However, DG method is still difficult for solving diffusion equations, because it is not easy to define numerical fluxes about the diffusion terms. Subsequently, [19] researched the local discontinuous Galerkin (LDG) method to deal with this problem. The main idea of LDG method is to change the original equation into a first-order system by introducing some auxiliary variables, which may result in high computational expense. To avoid this drawback, the hybridizable discontinuous Galerkin (HDG) method and the direct discontinuous Galerkin (DDG) method are proposed. So far as we know, [20] studied the HDG method for the spatial discretization of time fractional diffusion equation. The DDG method has also been applied in space approximation for TFPDE with the initial and boundary conditions, including time fractional reaction-diffusion problem with periodic boundary [21], time fractional reaction-diffusion problem with homogeneous Dirichlet conditions [22] and time fractional diffusion problem with Robin boundary [23]. It is worth noting that, DDG method can compel the weak form directly of PDE into the DG function space for the test functions and the numerical solutions, see [24, 25]. Since no new variables are introduced in the weak formulation of the problem, the DDG method has a practical advantage over LDG method. Thus, we will utilize the DDG method for the spatial discretization of (1.1) with FDBC.

Now we turn to consider the time discretization. L1 formula is a primary numerical formula to approximate time fractional derivatives with accuracy $2 - \alpha$ when the order of fractional derivative is α $(0 < \alpha < 1)$ [26, 27]. Actually, L1 formula is obtained by applying a piecewise linear interpolation at every small interval for the integrand. This guides us to replace linear interpolation with another higher-order interpolation to improve the accuracy. In the following literatures, several numerical Caputo formulae with higher precision are given, for example, L1-2 scheme [28] and L2-1_σ formula [29] which used the piecewise quadratic polynomial interpolation. The numerical solutions converge temporally with order 2 for subdiffusion equation (1.1a) with