INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 6, Pages 739–771

doi: 10.4208/ijnam2023-1033

A SHARP α-ROBUST $L1$ SCHEME ON GRADED MESHES FOR TWO-DIMENSIONAL TIME TEMPERED FRACTIONAL FOKKER-PLANCK EQUATION

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Abstract. In this paper, we are concerned with the numerical solution for the two-dimensional time fractional Fokker-Planck equation with the tempered fractional derivative of order α. Although some of its variants are considered in many recent numerical analysis works, there are still some significant differences. Here we first provide the regularity estimates of the solution. Then a modified $L1$ scheme inspired by the middle rectangle quadrature formula on graded meshes is employed to compensate for the singularity of the solution at $t \to 0^+$, while the five-point difference scheme is used in space. Stability and convergence are proved in the sense of L^∞ norm, getting a sharp error estimate $\mathscr{O}(\tau^{\min\{2-\alpha, r\alpha\}})$ on graded meshes. Furthermore, the constant multipliers in the analysis do not blow up as the order of Caputo fractional derivative α approaches the classical value of 1. Finally, we perform the numerical experiments to verify the effectiveness and convergence orders of the presented schemes.

Key words. Fractional diffusion equation, weak singularity, middle rectangle quadrature formula, modified $L1$ scheme, five-point difference scheme, graded mesh, α-robust.

1. Introduction

Anomalous diffusion with mean squared displacement (MSD) $\langle x^2(t)\rangle \simeq t^\alpha$, including subdiffusion and superdiffusion, is ubiquitously observed in a wide range of complex systems [1, 2, 3], and its anomalous diffusion exponent differs from the value $\alpha = 1$ of Brownian motion. Subdiffusion with $0 < \alpha < 1$ often occurs in cytoplasm of biological cells [4], amorphous semiconductors [5], or in hydrology [6]. Superdiffusion with $\alpha > 1$ is observed in some active systems such as molecular motor transport in cells [7] or in turbulence [8]. The continuous time random walk (CTRW) is one of the central stochastic models for both regimes of anomalous diffusion, which is based on two identically distributed random variables of the waiting times τ between any two jumps and the single jump lengths x. In fact, based on the fractional Fourier law and conservation law, the fractional diffusion equations (FDEs) can be also derived.

In this paper, we consider the generalized two-dimensional time fractional Fokker-Planck equation [9, 10]

$$\frac{\partial}{\partial t}u(x,y,t) = \frac{\partial}{\partial t}\int_0^t K(t-s,\mu)\Delta u(x,y,s)\,\mathrm{d}s, \tag{1}$$

which is derived from a CTRW model with the tempered α-stable waiting times. Here the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is solved in a rectangular domain $\Omega = (0, L_1) \times (0, L_2)$ and the Laplace transform of the memory kernel is given by $\widehat{K}(\lambda,\mu) = \frac{1}{(\lambda+\mu)^\alpha - \mu^\alpha}$ with the tempering index $\mu > 0$ and stability index $0 < \alpha < 1$. In fact, Ref. [10] indicates that the second moment of the CTRW model corresponding to (1) is $\langle x^2(t)\rangle \simeq t^\alpha$ as $t \to 0$, while $\langle x^2(t)\rangle \simeq t$ as $t \to \infty$. We first

Received by the editors on January 26, 2023 and, accepted on August 22, 2023.
2000 *Mathematics Subject Classification.* 65M06, 65M12.
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transform Eq. (1) into an equivalent Eq. (A.1) (see Appendix A). Without loss of generality, we consider Eq. (A.1) with a source term $f(x, y, t)$. Then under initial condition and homogeneous Dirichlet boundary condition, we discuss the time fractional Fokker-Planck initial-boundary value problem:

$$\begin{cases} \partial_t^{\alpha,\mu} u(x,y,t) - \Delta u(x,y,t) = f(x,y,t), & (x,y,t) \in Q := \Omega \times (0,T], \\ u(x,y,t) = 0, & (x,y,t) \in \partial\Omega \times (0,T], \\ u(x,y,0) = \phi(x,y), & (x,y) \in \bar{\Omega}, \end{cases} \tag{2}$$

where Ω is a bounded domain, $\bar{Q} := \bar{\Omega} \times [0, T]$, and f and ϕ are given functions; the time fractional derivative $\partial_t^{\alpha,\mu} u(x, y, t)$ is the tempered Caputo fractional derivative of order α, defined by

$$\partial_t^{\alpha,\mu} u(x,y,t) = \int_0^t w_\mu^\alpha(t-s) \frac{\partial}{\partial s} u(x,y,s)\,\mathrm{d}s \tag{3}$$

with

$$w_\mu^\alpha(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_t^\infty e^{-\mu s} s^{-1-\alpha}\,\mathrm{d}s. \tag{4}$$

Here $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t}\,\mathrm{d}t$ is the Gamma function. Clearly, when $\mu = 0$, $\partial_t^{\alpha,\mu} u$ is just the classical Caputo fractional derivative of order α.

However, since the nonlocal properties of fractional operators, it is more challenging or sometimes even impossible to obtain the analytical solutions of FDEs, or the obtained analytical solutions are less practicable (expressed by the transcendental functions or infinite series). Efficiently solving FDEs naturally has always been an urgent topic. So far there have been many works, including the finite difference method, finite element method, spectral method, and so on [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In particular, for the fractional derivatives in Caputo sense, the $L1$-type scheme [20, 21] and k-step backward difference formulae [15, 16, 22, 23] on uniform meshes are two popular and predominant discretization techniques. To the best of our knowledge, the smoothness of all the known data of (2) does not imply smoothness of the solution u in the closed domain $\bar{Q}$ [24, 25, 26]. In the early research, most papers ignored the possible presence of an initial layer in the solution at $t = 0$, and the corresponding convergence analyses make an unrealistic assumption that u is smooth in the closed domain $\bar{Q}$. Until later, the nonuniform time meshes were successfully employed to compensate for the singularity of the solution at $t \to 0^+$ [28, 29, 30, 31], which are flexible and reasonably convenient for practical implementation. Such graded meshes were originally used in the context of Volterra integral equations with weakly singular kernels [32, 33]. In particular, the $L1$ schemes on graded meshes for discretizing the fractional derivatives in Caputo sense with the optimal rate of convergence $\mathscr{O}(\tau^{\min\{2-\alpha, r\alpha\}})$ have been detailedly discussed in [17, 28]. However, it seems not easy to extend to the tempered Caputo fractional derivative $\partial_t^{\alpha,\mu} u(x, y, t)$ in our Fokker-Planck equation (2) because the kernel function $w_\mu^\alpha(t)$ is an improper integral. In this paper, a modified $L1$ scheme is designed to discretize the tempered Caputo fractional derivative $\partial_t^{\alpha,\mu} u(x, y, t)$, which seems to be the first time to be considered, and the classical five-point finite difference scheme is used to approximate Δu. After verifying the regularity of the solution to (2), a precise stability result and sharp α-robust error estimate $\mathscr{O}(\tau^{\min\{2-\alpha, r\alpha\}} + h^2)$ are obtained.

The structure of the paper is as follows. In Section 2, the regularity of the solution u of (2) is investigated and the bounds of those derivatives of u are derived, which are needed for the subsequent numerical analyses. In Section 3, the kernel