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Extremal Functions for an Improved Trudinger-Moser Inequality Involving L^p-Norm in $\mathbb{R}^n$

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Abstract. Let $W^{1,n}(\mathbb{R}^n)$ be the standard Sobolev space. For any $\tau>0$ and $p>n>2$, we denote

$$\lambda_{n,p}=\inf_{u\in W^{1,n}(\mathbb{R}^n),u\not\equiv 0}\frac{\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)\mathrm{d}x}{\left(\int_{\mathbb{R}^n}|u|^p\mathrm{d}x\right)^{\frac{n}{p}}}.$$

Define a norm in $W^{1,n}(\mathbb{R}^n)$ by

$$\|u\|_{n,p}=\left(\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)\mathrm{d}x-\alpha\Big(\int_{\mathbb{R}^n}|u|^p\mathrm{d}x\Big)^{\frac{n}{p}}\right)^{\frac{1}{n}},$$

where $0\le\alpha<\lambda_{n,p}$. Using a rearrangement argument and blow-up analysis, we will prove

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\|u\|_{n,p}\le 1}\int_{\mathbb{R}^n}\left(e^{\alpha_n|u|^{\frac{n}{n-1}}}-\sum_{j=0}^{n-1}\frac{\alpha_n^j|u|^{\frac{jn}{n-1}}}{j!}\right)\mathrm{d}x$$

can be attained by some function $u_0\in W^{1,n}(\mathbb{R}^n)\cap C^1(\mathbb{R}^n)$ with $\|u_0\|_{n,p}=1$, here $\alpha_n=n\omega_{n-1}^{\frac{1}{n-1}}$ and ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$.

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1 Introduction

Let $n\geq 2$, and denote $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. The famous Trudinger-Moser inequality [1–5] states that, for a bounded domain $\Omega\subset\mathbb{R}^n$ and $0<\gamma\leq\alpha_n$,

$$\sup_{u\in W_0^{1,n}(\Omega),\int_\Omega |\nabla u|^n dx\leq 1}\int_\Omega e^{\gamma|u|^{\frac{n}{n-1}}}\mathrm{d}x<\infty. \tag{1.1}$$

If $\gamma>\alpha_n$, the integrals in (1.1) are still finite, but the supremum is infinity.

One of the interesting questions about Trudinger-Moser inequalities is whether extremal function exists or not. The first result in this direction was obtained by Carleson-Chang [6] in the case that Ω is a unit disk in $\mathbb{R}^n$, then by Struwe [7] when Ω is a close to the ball in the sense of measure, by Flucher [8] for any bounded smooth domain in $\mathbb{R}^2$, and by Lin [9] to an arbitrary domain in $\mathbb{R}^n$.

The Trudinger-Moser inequality (1.1) was extended by Cao [10], Panda [11], do Ó [12], Ruf [13], and Li-Ruf [14] to the entire Euclidean space $\mathbb{R}^n (n\geq 2)$. Precisely, for any $\gamma\leq\alpha_n$,

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\int_{\mathbb{R}^n}(|\nabla u|^n+|u|^n)\mathrm{d}x\leq 1}\int_{\mathbb{R}^n}\left(e^{\gamma|u|^{\frac{n}{n-1}}}-\sum_{j=0}^{n-2}\frac{\gamma^j|u|^{\frac{nj}{n-1}}}{j!}\right)\mathrm{d}x<\infty. \tag{1.2}$$

Adimurthi-Yang [15] generalized (1.2) to a singular version. That is, for all $\tau>0$, $n\geq 2$, $0<\beta<1$ and $0<\eta\leq 1-\beta$, one has

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)\mathrm{d}x\leq 1}\int_{\mathbb{R}^n}\frac{e^{\alpha_n\eta|u|^{\frac{n}{n-1}}}-\sum_{j=0}^{n-2}\frac{\alpha_n^j\eta^j|u|^{\frac{jn}{n-1}}}{j!}}{|x|^{n\beta}}\mathrm{d}x<\infty. \tag{1.3}$$

Obviously, for all $\tau\in(0,+\infty)$, the norms $(\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)\mathrm{d}x)^{\frac{1}{n}}$ are equivalent to the standard norms on $W^{1,n}(\mathbb{R}^n)$. Then Li-Yang [16] obtained the existence of extremal functions for (1.3) using blow-up analysis. Later, (1.3) was extended by Li [17] to the following modified form. Let $p>n\geq 2$ and

$$\lambda_{n,p}=\inf_{u\in W^{1,n}(\mathbb{R}^n),u\not\equiv 0}\frac{\int_{\mathbb{R}^n}(|\nabla u|^n+\tau|u|^n)\mathrm{d}x}{\left(\int_{\mathbb{R}^n}|u|^p\mathrm{d}x\right)^{n/p}}. \tag{1.4}$$

For $0<\beta<1$ and $0\leq\alpha<\lambda_{n,p}$, the supremum

$$\sup_{u\in W^{1,n}(\mathbb{R}^n),\|u\|_{n,p}\leq 1}\int_{\mathbb{R}^n}\frac{e^{\alpha_n(1-\beta)|u|^{\frac{n}{n-1}}}-\sum_{j=0}^{n-2}\frac{\alpha_n^j(1-\beta)^j|u|^{\frac{jn}{n-1}}}{j!}}{|x|^{n\beta}}\mathrm{d}x \tag{1.5}$$