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Free Boundaries Problem for a Class of Parabolic Type Chemotaxis Model

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Abstract. In this paper, we are interested in a free boundary problem for a chemotaxis model with double free boundaries. We use contraction mapping principle and operator-theoretic approach to establish local solvability of a chemotaxis system in 1-Dimensional domain with non-constant coefficient free boundaries.

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1 Introduction

In this paper, we consider a free boudary problem for a chemotaxis model with double free boundaries. The model reads as follows

$$\begin{cases} u_t(x,t)=(u_x(x,t)-u(x,t)v_x(x,t))_x, & g(t)<x<h(t),\ 0<t<T,\\ u(x,t)=0, & -1<x<g(t),\ h(t)<x<1,\ 0<t<T,\\ u_x(g(t),t)+k_1(g(t),t)u(g(t),t)=0, & 0<t<T,\\ g_t(t)=k_1(g(t),t)+v_x(g(t),t), & 0<t<T,\\ u_x(h(t),t)+k_2(h(t),t)u(h(t),t)=0, & 0<t<T,\\ h_t(t)=k_2(h(t),t)+v_x(h(t),t), & 0<t<T,\\ -g(0)=h(0)=b, u(x,0)=u_0(x), & -b<x<b,\\ v_t(x,t)=v_{xx}(x,t)+u(x,t)-v(x,t), & -1<x<1,\ 0<t<T,\\ v_x(-1,t)=0, & 0<t<T,\\ v_x(1,t)=0, & 0<t<T,\\ v(x,0)=v_0(x), & -1<x<1, \end{cases} \tag{1.1}$$

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where

- $u=u(x,t)$ is an unknown function of $(x,t)\in(g(t),h(t))\times(0,T)$ and it stands for the density of cellular slime molds. In other words, the density $u(x,t)$ occupies the domain $(g(t),h(t))$, an open subset of $(-1,1)$, in time t and $u(x,t)=0$ in the outside of $(g(t),h(t))$;
- $v=v(x,t)$ is an unknown function of $(x,t)\in(-1,1)\times(0,T)$ and it stands for the concentration of chemical substances secreted by the slime molds;
- $k_1(x,t)$, $k_2(x,t)$ are given continuous functions which satisfy the Lipschitz condition on x, namely there exists $L>0$, such that

$$|k_i(x_1,t)-k_i(x_2,t)|\leqslant L|x_1-x_2|,\quad i=1,2, \tag{1.2}$$

  for any $t\in[0,+\infty)$. Also, $k_1(x,t)$, $k_2(x,t)$ are bounded on $t\in[0,+\infty)$. In other words, there exists $C>0$ which may depend on x, such that

$$|k_i(x,t)|\leqslant C,\quad i=1,2; \tag{1.3}$$

- $g(t)$, $h(t)$ are two unknown moving boundaries;
- $b\in(0,1)$ is a given number.

For general smooth domain Ω, the system (1.1) is based on the well-known chemotaxis model with fixed boundary

$$\begin{cases} u_t=\nabla(\nabla u-u\nabla v), & \text{in } \Omega\times(0,T),\\ v_t=\Delta v-v+u, & \text{in } \Omega\times(0,T),\\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n}=0, & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & \text{in } \Omega,\\ v(x,0)=u_0(x), & \text{in } \Omega, \end{cases} \tag{1.4}$$

introduced by E. F. Keller and L. A. Segel [1]. The problem (1.4) is intensively studied by many authors (see for instance [2–8]). The initial functions $u_0\in C^0(\overline{\Omega})$ and $v_0\in C^1(\overline{\Omega})$ are assumed to be nonnegative. Within this framework, classical results state that

- if n=1 then all solutions of (1.4) are global in time and bounded [9];
- if n=2 then
  - in the case $\int_\Omega u_0(x)\mathrm{d}x<4\pi$, the solution will be global and bounded [10, 11], whereas