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ODE-Based Multistep Schemes for Backward Stochastic Differential Equations

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Abstract. In this paper, we explore a new approach to design and analyze numerical schemes for backward stochastic differential equations (BSDEs). By the nonlinear Feynman-Kac formula, we reformulate the BSDE into a pair of reference ordinary differential equations (ODEs), which can be directly discretized by many standard ODE solvers, yielding the corresponding numerical schemes for BSDEs. In particular, by applying strong stability preserving (SSP) time discretizations to the reference ODEs, we can propose new SSP multistep schemes for BSDEs. Theoretical analyses are rigorously performed to prove the consistency, stability and convergency of the proposed SSP multistep schemes. Numerical experiments are further carried out to verify our theoretical results and the capacity of the proposed SSP multistep schemes for solving complex associated problems.

AMS subject classifications: 65C30, 60H35, 65C20

Key words: Backward stochastic differential equation, parabolic partial differential equation, strong stability preserving, linear multistep scheme, high order discretization.

1. Introduction

In this work, we are concerned with numerical solutions of the Markovian backward stochastic differential equations (BSDEs) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T], \tag{1.1}$$

where $X_t = X_0 + \sigma W_t$, $T > 0$ is the deterministic terminal time, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered complete probability space with $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ being the natural filtration of the

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standard q-dimensional Brownian motion $W=(W_t)_{0\le t\le T}$. The matrix $\sigma\in\mathbb{R}^{d\times q}$ is the diffusion coefficient, $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^p\times\mathbb{R}^{p\times q}\to\mathbb{R}^p$ and $\xi=\varphi(X_T)$ with $\varphi:\mathbb{R}^d\to\mathbb{R}^p$ are the generator and the terminal condition of the BSDE, respectively. The stochastic integral with respect to W_s is of the Itô type. The pair of processes (Y,Z) is called an L^2-adapted solution of the BSDE (1.1) if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies (1.1).

In 1990, Pardoux and Peng [28] first proved the existence and uniqueness of the solutions of nonlinear BSDEs. In 1991, Peng [30] further put forward the nonlinear Feynman-Kac formula, which established a deep connection between the BSDE (1.1) and the parabolic partial differential equation (PDE), that is, under some regularity conditions, the solution (Y,Z) of (1.1) can be expressed as

$$Y_t=u(t,X_t),\quad Z_t=\nabla_x u(t,X_t),\quad t\in[0,T), \tag{1.2}$$

where $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}^p$ is the classical solution to the following PDEs:

$$\begin{aligned}&\left(\partial_t+\frac{1}{2}\sum_{i,j=1}^{d}\sum_{l=1}^{q}\sigma_{il}\sigma_{jl}\partial^2_{x_ix_j}\right)u(t,x)\\&\quad+f\big(t,x,u(t,x),\nabla_x u(t,x)\sigma\big)=0,\quad (t,x)\in[0,T)\times\mathbb{R}^d\end{aligned} \tag{1.3}$$

with the terminal condition $u(T,x)=\varphi(x)$ for $x\in\mathbb{R}^d$. The representation (1.2) is important in applications, which allows us to design numerical methods for parabolic PDEs (1.3) by solving the associated BSDEs (1.1) and vice versa.

The analytic solutions to BSDEs is usually unavailable, and thus numerical methods for solving BSDEs are highly desired. In recent years, great efforts have been made for designing efficient numerical schemes for BSDEs. There are two main type of schemes: the first type is based on numerical solution of a parabolic PDE which is related to the BSDE [9, 26, 27], and the second type of schemes focus on discretizing BSDEs directly [1–3, 8, 18, 25, 31, 34, 41]. For the second type of schemes, popular temporal discretizations include Euler-type methods [11, 12, 40], θ-schemes [41, 44], Runge-Kutta schemes [6], and multistep schemes [5, 42, 45, 46], etc.

Most of the aforementioned temporal discretizations have their prototypical counterparts designed for ordinary differential equations (ODEs). However, it is still a non-trivial task to extend the ODEs' numerical methods to the ones for BSDEs. Actually, in conventional approaches, e.g., [34, 41, 42, 44, 45], etc, numerical approximations to the solutions (Y,Z) are achieved by discretizing a pair of reference equations deduced from (1.1), of which the typical form are that

$$\mathbb{E}_t[Y_s]=\mathbb{E}_t[\xi]+\int_s^T\mathbb{E}_t[f(r,X_r,Y_r,Z_r)]\mathrm{d}r, \tag{1.4}$$

$$\int_t^s\mathbb{E}_t[Z_r]\mathrm{d}r=\mathbb{E}_t[Y_s\Delta W_{t,s}^{\top}]+\int_t^s\mathbb{E}_t[f(r,X_r,Y_r,Z_r)\Delta W_{t,r}^{\top}]\mathrm{d}r, \tag{1.5}$$