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Projection Improved SPAI Preconditioner for FGMRES

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Abstract. Krylov subspace methods are widely used for solving sparse linear algebraic equations, but they rely heavily on preconditioners, and it is difficult to find an effective preconditioner that is efficient and stable for all problems. In this paper, a novel projection strategy including the orthogonal and the oblique projection is proposed to improve the preconditioner, which can enhance the efficiency and stability of iteration. The proposed strategy can be considered as a minimization process, where the orthogonal projection minimizes the energy norm of error and the oblique projection minimizes the 2-norm of the residual, meanwhile they can be regarded as approaches to correct the approximation by solving low-rank inverse of the matrices. The strategy is a wide-ranging approach and provides a way to transform the constant preconditioner into a variable one. The paper discusses in detail the projection strategy for sparse approximate inverse (SPAI) preconditioner applied to flexible GMRES and conducts the numerical test for problems from different applications. The results show that the proposed projection strategy can significantly improve the solving process, especially for some non-converging and slowly convergence systems.

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Key words: Generalized minimal residual, preconditioner, sparse approximate inverse, oblique projection, orthogonal projection.

1. Introduction

Solving sparse linear algebraic equations is extensively used for many numerical simulations, which greatly affects the efficiency of these applications. The main methods for solving large systems of sparse linear equations include the direct and iterative methods. For some relatively small-scale matrices, the direct methods can work well. However, when the scale exceeds a certain level, usually up to a few millions of un-

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knowns, the direct methods may be no longer applicable due to the dramatic increase in memory consumption and computational effort [6]. In such a situation, the iterative methods become the only choice. Among these methods, the Krylov subspace method is the most widely used method, especially for systems arising from discretization of partial differential equations. Krylov subspace methods include the conjugate gradient algorithm (CG) for solving symmetric positive definite matrices, the generalised minimal residual (GMRES) method and the biconjugate gradient stabilized (BICGSTAB) method for solving general non-symmetric problems, etc. [13]. In order to accelerate the convergence of Krylov subspace methods, usually, a preconditioning method is combinedly used.

There exist many kinds of preconditioning methods, including the Jacobi, the Gauss-Seidel methods, etc., and their other relaxation versions such as the Jacobi over relaxation (JOR) method [1] and the successive over relaxation (SOR) method [5]. There are also some simple but more efficient incomplete decomposition preconditioners, for instance incomplete LU decomposition (ILU) [13] and its variants with different filling level as ILU(k), ILUP [9], etc., and ILUT [12] with thresholding value. Another class is the sparse approximate inverse (SPAI) preconditioner [4]. There are some multi-level preconditioning methods as well, but their implementation may be more complex, and the representative of these is the algebraic multigrid (AMG) [15] method. However, each of these methods has its advantages and disadvantages. For kinds of problem, some preconditioning methods are very effective, but for many other problems they do not work well.

Recently some methods have been developed to accelerate the solution of classical iteration, such as Anderson Richardson iterative method [10, 16] and adaptive relaxation algorithm [17], and they are even more efficient than Krylov algorithm in solving some kinds of system. Inspired by this, a projection strategy for improving usual preconditioning methods is proposed in the paper. Through increasing the number of preconditioner iterations, the strategy then utilizes these historical approximate solutions and residuals to obtain a better approximation so as to accelerate the solving process. We adopt two projection methods, the orthogonal projection method and the oblique projection method, both of them use the historical information from the iterative process to construct the projection subspace. The projection process can be understood as a process of solving a low-rank inverse [7], or it can be regarded as a minimization process. The strategy provides a way to transform the constant preconditioner to a variable preconditioner, and it is realized for sparse approximate inverse preconditioner in the paper. The solvers used are GMRES method and flexible GMRES (FGMRES) method [12]. FGMRES is designed for a variable preconditioner, which means that its preconditioning operator can be changed during the iteration progress.

In the following, an improved preconditioner will be described in detail and a simple theoretical analysis will be given. The effectiveness of the algorithm will be illustrated through numerical experiments for different fields of problem. The algorithm is implemented through AMGCL [3], which is a C++ AMG library containing only header files. To illustrate the parallel efficiency of the algorithm, the CUDA backend imple-