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Space-Time Decomposition of Kalman Filter

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Abstract. We present an innovative interpretation of Kalman filter (KF) combining the ideas of Schwarz domain decomposition (DD) and parallel in time (PinT) approaches. Thereafter we call it DD-KF. In contrast to standard DD approaches which are already incorporated in KF and other state estimation models, implementing a straightforward data parallelism inside the loop over time, DD-KF ab-initio partitions the whole model, including filter equations and dynamic model along both space and time directions/steps. As a consequence, we get local KFs reproducing the original filter at smaller dimensions on local domains. Also, sub problems could be solved in parallel. In order to enforce the matching of local solutions on overlapping regions, and then to achieve the same global solution of KF, local KFs are slightly modified by adding a correction term keeping track of contributions of adjacent subdomains to overlapping regions. Such a correction term balances localization errors along overlapping regions, acting as a regularization constraint on local solutions. Furthermore, such a localization excludes remote observations from each analyzed location improving the conditioning of the error covariance matrices. As dynamic model we consider shallow water equations which can be regarded a consistent tool to get a proof of concept of the reliability assessment of DD-KF in monitoring and forecasting of weather systems and ocean currents.

AMS subject classifications: 35Q93, 49M27, 49M41, 65K15, 65M32

Key words: Data assimilation, Kalman filter, domain decomposition, filter localization, model reduction, numerical algorithm.

1. Introduction

In the present paper we focus on Kalman filter which is one of the most important and common state estimation algorithms solving data assimilation (DA) problems [24, 25]. Besides its employment in validation of mathematical models used in meteorology, climatology, geophysics, geology and hydrology [50], it has become a main component in visual tracking of moving object in image processing or autonomous vehicles tracking with GPS in satellite navigation systems, or even in applications of non physical systems

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such as financial markets [3, 20, 26, 50]. Its main strength is the simple derivation of the algorithm using the filter recursive property: measurements/observations are processed step by step as they arrive allowing to correct the given data. A recent review of KF and its applications is given in [28].

KF time complexity requires very large computational burden in concrete scenarios [33, 46, 48]. Hence, several variants/approximation have been proposed to reduce the computational complexity and to allow KF deployment in real time. These are designed on the basis of a reduction in the order of the model [16,18,42,49], or they are based on ensemble Kalman filter (EnKF) which approximates KF by representing the distribution of the state with an ensemble of draws from that distribution. A prediction of the error at a future time is computed by integrating each ensemble state independently by the model [15]. EnKF can be used for nonlinear dynamics settings and they are more suited to large scale real world problems than the KF, so it is more viable for use within operational DA [38]. Nevertheless, EnKF methods are hindered by a reduction in the size of the ensemble due to the computational requirements of maintaining a large ensemble [1, 17]. In applications, most efforts to deal with the related problems of computational effort and sampling error in ensemble estimation have focused on using variations on the concept of localization [51].

A typical approach for solving computationally intensive problems – which is oriented to exploit parallel computing – is based on Schwarz domain decomposition (DD) techniques. DD methods are well-established strategies that has been used with great success for a wide variety of problems. Review papers, both for mathematical and computational analysis are for instance [34, 40, 43, 44]. Concerning DA problems, DD is usually performed by discretizing the objective function and the model to build a discrete Lagrangian [9]. At the heart of these so called all-at-once approaches, lies solution of a very large linear system (the Karush-Kuhn-Tucker (KKT) system) which is decomposed according to the discretization grid. A common drawback of such parallel algorithms is their limited scalability, due to the fact that parallelism is achieved adapting the most computationally demanding tasks for parallel execution. Concerning KF this approach leads to the straightforward data parallelism inside the loop over timesteps. Amdhal's law clearly applied in these situations because the computational cost of the components that are not parallelized – or in case of KF a data synchronization at each step – provides a lower bound on the execution time of the parallel algorithm. More generally if a fraction r of our serial program remains unparallelized, then Amdahl's law says we can not get a speedup better than $1/r$. Thus even if r is quite small, say $1/100$, and we have a system with thousands of cores, we cannot get a speedup better than 100.

Here we present the mathematical framework of an innovative approach which turns to be appropriate for using state estimation problems described by KF and governed by a dynamic model such as partial differential equations (PDEs). This approach relies on the ideas of Schwarz DD [44] and parallel in time approaches [30] fully revised in a linear algebra setting. Schwarz methods use as boundary conditions of the local PDE-model the approximation of the numerical solution computed on the interfaces