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ALIKHANOV LINEARIZED GALERKIN FINITE ELEMENT METHODS FOR NONLINEAR TIME-FRACTIONAL SCHRÖDINGER EQUATIONS*

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Abstract

We present Alikhanov linearized Galerkin methods for solving the nonlinear time fractional Schrödinger equations. Unconditionally optimal estimates of the fully-discrete scheme are obtained by using the fractional time-spatial splitting argument. The convergence results indicate that the error estimates hold without any spatial-temporal stepsize restrictions. Numerical experiments are done to verify the theoretical results.

Mathematics subject classification: 65M60, 65M12, 65M15.
Key words: Fractional Grönwall type inequality, Nonlinear time-fractional Schrödinger equation, Error analysis.

1. Introduction

We consider the Alikhanov finite element method (FEM) for solving the following nonlinear time fractional Schrödinger equations (TFSEs) [29]:

$$\begin{cases} i^C D_t^\alpha u + \Delta u + f\big(|u|^2\big)u = 0, & (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}, t), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}, t) = 0, & (\mathbf{x}, t) \in \partial\Omega \times [0, T], \end{cases} \tag{1.1}$$

where $i = \sqrt{-1}, \Omega \in \mathcal{R}^d$, $d = 1, 2, 3$, and $f \in C^3(\mathcal{R})$ is a nonlinear function, $u(\mathbf{x}, t)$ is a complex-valued function. Here ${}^C D_t^\alpha u$ denotes the Captuo fractional derivative, which is defined as

$${}^C D_t^\alpha u(x, t) = \int_0^t w_{1-\alpha}(t - s) \frac{\partial u(x, s)}{\partial s} ds,$$

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where $w_\beta(t) = t^{\beta-1}/\Gamma(\beta)$ and $\Gamma(\cdot)$ is the common Gamma function. Fractional Schrödinger equations were investigated extensively. For example, Laskin [16, 17] proposed the fractional Schrödinger equations by using the Feynman path integrals instead of Lévy ones. In 2004, Naber [29] pointed out that one could obtain a time fractional Schrödinger equation when non-Markovian evolution was considered. In 2006, Xu and Guo [7] studied some physical evolutions of fractional Schrödinger equation. Tofighi [32] considered the probability structure of TFSEs. More details can be found in [1, 9, 18, 28, 30].

In the past several years, TFSEs were numerically investigated by using different algorithms, including finite difference methods [5, 11, 14, 15, 26, 33], finite element methods [24], spectral methods [3, 8, 35], local discontinuous Galerkin methods [34] and Krylov projection method [6] and so on [2, 10, 12]. Most convergence results were obtained with certain time-step restrictions dependent on the spatial mesh sizes. In order to remove such restrictions, Li *et al.* [21] introduced the fractional temporal-spatial splitting argument and obtained unconditionally optimal L^2-error estimates for problem (1.1). The time-discretization in the paper is done by L1 scheme. And the convergence order of the scheme is $2-\alpha$ in temporal direction if the exact solutions satisfy $u \in C^2([0,T]; L^2(\Omega))$. The regularity of the problems is not considered in the paper.

In this paper, we propose a linearized fully-discrete numerical scheme for solving problem (1.1), taking the initial singularities into account. The temporal discretization is done by applying the Alikhanov scheme on graded meshes and the extrapolation method. The spatial discretization is done by using the r-degree Galerkin FEM. It is shown that the convergence order in L^2-norm of the fully-discrete scheme can be of 2 in the temporal direction and of $r+1$ in the spatial direction. Such error estimates hold without any spatial-temporal stepsize restrictions. The key to the proof is the so-called temporal-spatial splitting argument, which is firstly proposed by Li and Sun [19, 20]. This technique has a successful application in the time-dependent problems [21–23, 37]. We introduce the approach in our proof and obtain the unconditionally convergent results for the time fractional problems in the complex spaces.

The rest of this paper is organized as follows. In Section 2, Alikhanov linearized Galerkin FEM is established for solving problem (1.1), and our main results are also presented. In Section 3, a rigorous analysis of our results is obtained by applying discrete fractional Grönwall type inequality. In Section 4, numerical examples are given to confirm our theoretical results. Finally, some conclusions are drawn in Section 5.

Throughout this paper, C_ν and C_f denote two positive constants, not always the same in different occasions, which dependent on the given information but independent of temporal and spatial stepsizes.

2. The Alikhanov Galerkin FEM

Following the standard FEM discretization [31], let $\mathcal{T}_h$ be a subdivision of Ω into triangles T_k in $\mathcal{R}^1, \mathcal{R}^2$ or tetrahedra in $\mathcal{R}^3$ and $h = \max_{T_K \in \mathcal{T}_h}\{\text{diam } T_K\}$ be the mesh size. The finite-dimensional subspace of $H_0^1(\Omega)$ is named V_h. It is comprised by continuous piecewise polynomial $\{\phi_j\}_{j=1}^M$ whose order is r $(r \geq 1)$ on $\mathcal{T}_h$. Let $\tau_n = t_{n+1} - t_n$ be time step. Denote $t_n = T(n/N)^\delta, 0 \leq n \leq N, \delta \geq 1$, $t_{n-\alpha/2} = (1-\alpha/2)t_n + (\alpha/2)t_{n-1}$, where N is a given integer and $u^m = u(x, t_m)$. For a set of functions $\{\omega^n\}$, we define

$$\begin{aligned} \omega^{n,\alpha} &= \left(1 - \frac{\alpha}{2}\right)\omega^n + \frac{\alpha}{2}\omega^{n-1}, && 1 \leq n \leq N, \\ \hat{\omega}^n &= \left(2 - \frac{\alpha}{2}\right)\omega^{n-1} - \left(1 - \frac{\alpha}{2}\right)\omega^{n-2}, && n \geq 2. \end{aligned} \tag{2.1}$$