Ann. Appl. Math.
doi: 10.4208/aam.OA-2023-0027

Vol. **39**, No. 4, pp. 406-428
November 2023

Connections between Operator-Splitting Methods and Deep Neural Networks with Applications in Image Segmentation

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Received 12 July 2023; Accepted (in revised version) 16 October 2023

Dedicated to the memory of Professor Zhongci Shi

Abstract. Deep neural network is a powerful tool for many tasks. Understanding why it is so successful and providing a mathematical explanation is an important problem and has been one popular research direction in past years. In the literature of mathematical analysis of deep neural networks, a lot of works is dedicated to establishing representation theories. How to make connections between deep neural networks and mathematical algorithms is still under development. In this paper, we give an algorithmic explanation for deep neural networks, especially in their connections with operator splitting. We show that with certain splitting strategies, operator-splitting methods have the same structure as networks. Utilizing this connection and the Potts model for image segmentation, two networks inspired by operator-splitting methods are proposed. The two networks are essentially two operator-splitting algorithms solving the Potts model. Numerical experiments are presented to demonstrate the effectiveness of the proposed networks.

AMS subject classifications: 68U10, 94A08
Key words: Potts model, operator splitting, deep neural network, image segmentation.

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1 Introduction

In past decades, deep neural network has emerged as a very successful technique for various fields. It has demonstrated impressive performances in many tasks, such as image processing, object detection, and natural language processing. In some tasks, deep neural networks even outperform humans.

Due to great successes of neural networks, over the past several years, a lot of works has been devoted to the mathematical understanding of neural networks and explaining their success. Representation theories for function learning are studied in [4,17,38,46,66] for feedforward neural networks, in [70,71] for convolutional neural networks, and in [36,37,54] for convolutional residual networks. Recently, theoretical results for learning operators are developed in [5, 33, 44]. The works mentioned above show that as long as the network depth and width are sufficiently large, deep neural networks can approximate any function or operator within a certain class to arbitrary accuracy. These works focus on the existence of a good approximator with desired approximation error and use techniques from approximation theory. Recently, analysis of neural ordinary differential equations from the perspective of optimal control was conducted in [58]. In this paper, we investigate the power of neural networks from another perspective: the network structure. We will give an algorithmic explanation of neural networks, especially in their connections with operator-splitting algorithms.

The operator-splitting method is a class of powerful methods for numerically solving complicated problems. The general idea is to decompose a difficult problem into several subproblems which will be solved sequentially or simultaneously. Operator-splitting methods have been widely used on solving partial differential equations [24, 26, 47, 49, 64], image processing [21, 22, 40, 41], surface reconstruction [30], obstacle problem [43], inverse problem [25], and computational fluid dynamics [6, 52], etc. We refer readers to [28, 48] for some survey discussions.

Image segmentation is an important subject in many fields, such as medical imaging and object detection. Many mathematical models and algorithms have been developed for image segmentation [10, 11, 13–15, 31, 53, 63]. In [2, 68], the segmentation problem is formulated as a min cut or max flow problem. One important model for image segmentation is the Potts model, which was first proposed for statistical mechanics [56]. In fact, the well-known Chan-Vese model [15] is a special case of the Potts model. In [65], detailed explanations are given to show that the Potts model is equivalent to a continuous min cut and max flow problem [65]. Efficient algorithms for the Potts model are studied in [60, 68]. We suggest readers to survey [61] for a comprehensive discussion on the Potts model. Recently, many deep learning methods for image segmentation are also proposed [23, 57, 72].