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A Discontinuity and Cusp Capturing PINN for Stokes Interface Problems with Discontinuous Viscosity and Singular Forces

Yu-Hau Tseng[1] and Ming-Chih Lai[2,*]

[1] *Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 81148, Taiwan*

[2] *Department of Applied Mathematics, National Yang Ming Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, Taiwan*

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Dedicated to the memory of Professor Zhongci Shi

Abstract. In this paper, we present a discontinuity and cusp capturing physics-informed neural network (PINN) to solve Stokes equations with a piecewise-constant viscosity and singular force along an interface. We first reformulate the governing equations in each fluid domain separately and replace the singular force effect with the traction balance equation between solutions in two sides along the interface. Since the pressure is discontinuous and the velocity has discontinuous derivatives across the interface, we hereby use a network consisting of two fully-connected sub-networks that approximate the pressure and velocity, respectively. The two sub-networks share the same primary coordinate input arguments but with different augmented feature inputs. These two augmented inputs provide the interface information, so we assume that a level set function is given and its zero level set indicates the position of the interface. The pressure sub-network uses an indicator function as an augmented input to capture the function discontinuity, while the velocity sub-network uses a cusp-enforced level set function to capture the derivative discontinuities via the traction balance equation. We perform a series of numerical experiments to solve two- and three-dimensional Stokes interface problems and perform an accuracy comparison with the augmented immersed interface methods in literature. Our results indicate that even a shallow network with a moderate number of neurons and

*Corresponding author.
Email: `mclai@math.nctu.edu.tw` (M.-C. Lai)

sufficient training data points can achieve prediction accuracy comparable to that of immersed interface methods.

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1 Introduction

The incompressible two-phase flow problems have a wide range of applications in various scientific and engineering fields; see [7] and the references therein. Very often, this kind of problem involves solving Stokes equations with a discontinuous viscosity and singular force (such as surface tension force) along the fluid interface. For simplicity, we call this problem the Stokes interface problem hereafter. Solving Stokes interface problems numerically is known to be quite challenging in the scientific computing community, giving rise to several difficulties. One major difficulty is to handle the coupling between two adjacent fluids and the interfacial boundary conditions (arising from the singular force). Another difficulty comes from that the pressure is, in fact, discontinuous, and the partial derivatives of the velocity are also discontinuous across the interface. Thus, careful numerical treatments must be adopted for accurate discretizations near the interface, regardless of using the finite difference or finite element method.

In order to address the above issues, Peskin proposed the immersed boundary (IB) method [16], which introduces a regularized version of the discrete delta function to discretize the singular force in a Cartesian grid setting. The discontinuous viscosity can also be regularized by a smoothed version of the Heaviside function. In this framework, the problem becomes smooth so that regular finite difference discretizations such as the MAC scheme can be applied to solve the Stokes equations. However, it is known that the IB method achieves only first-order accuracy [12] for the solution variables. To improve the accuracy, an alternative approach is to solve the Stokes equations within each fluid domain separately and reformulate the singular force term into the traction balance equation on the interface. The above two formulations will be given in Section 2. In this paper, we shall introduce a neural network methodology to solve the problem based on the second formulation. Since the present goal here is to make comparisons with traditional methods such as immersed interface method (IIM, also based on the second formulation) from the