CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2022-0050

Vol. **4**, No. 4, pp. 758-775
December 2023

A Two-Level Simultaneous Orthogonal Matching Pursuit Algorithm for Simultaneous Sparse Approximation Problems

Nian Shao[1], Rui Zhang[2], Jie Sun[2] and Wenbin Chen[1,3,*]

[1] *School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R. China.*
[2] *Theory Lab, Central Research Institute, 2012 Labs, Huawei Technologies Co. Ltd., Hong Kong SAR, P.R. China.*
[3] *Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, P.R. China.*

Received 21 December 2022; Accepted 29 May 2023

Abstract. In this paper, we propose a two-level simultaneous orthogonal matching pursuit (TLSOMP) algorithm for simultaneous sparse approximation (SSA) problems. Most existing algorithms for SSA problems are directly generalized from the ones for the sparse approximation (SA) problems, for example, the simultaneous orthogonal matching pursuit (SOMP) method is generalized from the orthogonal matching pursuit (OMP) method. Our newly proposed algorithm is designed from another viewpoint. We first analyze the noiseless case and propose a selection algorithm. Motivated by the analysis and presuming noise as a perturbation, we extend the selection algorithm into a TLSOMP algorithm. This novel algorithm mainly uses the information from the subspace spanned by the multiple signals, which is not available in SA problems. Numerical experiments show the superiority of our TLSOMP algorithm over other traditional SSA solvers.

AMS subject classifications: 65F20, 65F22, 65F55

Key words: Simultaneous sparse approximation, two-level simultaneous orthogonal matching pursuit, subspace approximation.

---

1 Introduction

The sparse approximation problem, which aims at recovering a signal with the fewest linear combination of elements from a redundant dictionary [35], has been studied extensively [34], and it has various applications [5,12,16,28,31]. In mathematical formula, this

*Corresponding author. *Email addresses:* `nshao20@fudan.edu.cn` (N. Shao), `j.sun@huawei.com` (J. Sun), `zhangrui191@huawei.com` (R. Zhang), `wbchen@fudan.edu.cn` (W. Chen)

problem can be illustrated as

$$Y = AX + \mathcal{E}, \tag{1.1}$$

where A is the redundant dictionary, X is an unknown row-sparse matrix, i.e. only few rows of X are nonzero, and $\mathcal{E}$ is the noise. When X and Y are vectors, the problem is called sparse approximation (SA) problem, and while X and Y are matrices, we call it simultaneous sparse approximation problem.

Related work. There are many classical methods for SA problems, such as least squares support vector machines (LS-SVMs) method [33], orthogonal matching pursuit method [35], sparse Bayesian learning (SBL) method [43], iterative thresholding method [19], penalty decomposition method [25]. For the mathematical perspective of SA problems, which is also known as compressive sensing, we refer the readers to [11].

A natural extension of the SA problem is the SSA problem, which aims at finding jointly sparse representations of multiple signals. There are also lots of applications of SSA problems, such as [7, 14, 15, 26]. Most methods for SA problems have been generalized into the SSA problems, such as simultaneous orthogonal matching pursuit method [38], convex relaxation method [36], simultaneous sparse Bayesian learning (SSBL) method [44]. Other classical methods can be found in the survey [32]. Most of these methods handle the multiple signals by a direct way, and recently, a spectral pursuit method for the SSA problem was proposed in [40], which considers the spectral information of the multiple signals.

The key point of solving SSA problem is to use the multiple signals Y efficiently together. In this paper, we explore the problem in a subspace view, and the main technique for subspace approximation is the perturbation theory about singular value decomposition (SVD). The basic perturbation bound about the singular values of a matrix is the Weyl's inequality, which can be dated back to one century before [42], and it has a generalization of the Mirsky's inequality [30]. For the analysis of the subspace, it has a fruitful research history. To the best of our known, the first result in this area is the Wedin's theorem [41]. Due to the important role of SVD in numerical linear algebra (NLA), there are also lots of significant researches in NLA view, such as [21–23].

Contributions. In this paper, we propose a TLSOMP algorithm for the SSA problem.

The first level is finding a well-conditioned sub-dictionary A_* satisfying that $\text{span}\{Y\}$ is almost in $\text{span}\{A_*\}$. First, we obtain the space information of Y by computing the truncated SVD, and the row-sparsity s is estimated by the numerical rank of Y. Suppose U is an orthogonal basis of the left singular space corresponding to the largest s singular values of Y, and in order to extract some columns of A to approximate $\text{span}\{U\}$, we first compute the error of $\|a - UU^\top a\|$ for all column vectors a of A, which measures the distance between a and $\text{span}\{U\}$. After finding N_1 columns of A, whose distance errors are small, we select N_2 columns from them as A_* by solving a column subset selection problem (CSSP) [1, 9, 37], which can be solved approximately by rank revealing QR factorization (RRQR) method [3, 4, 17].