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Asymptotic Expansion Regularization for Inverse Source Problems in Two-Dimensional Singularly Perturbed Nonlinear Parabolic PDEs

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Abstract. In this paper, we develop an asymptotic expansion-regularization (AER) method for inverse source problems in two-dimensional nonlinear and nonstationary singularly perturbed partial differential equations (PDEs). The key idea of this approach is the use of the asymptotic-expansion theory, which allows us to determine the conditions for the existence and uniqueness of a solution to a given PDE with a sharp transition layer. As a by-product, we derive a simpler link equation between the source function and first-order asymptotic approximation of the measurable quantities, and based on that equation we propose an efficient inversion algorithm, AER, for inverse source problems. We prove that this simplification will not decrease the accuracy of the inversion result, especially for inverse problems with noisy data. Various numerical examples are provided to demonstrate the efficiency of our new approach.

AMS subject classifications: 65M32, 35C20, 35G31

Key words: Inverse source problem, singular perturbed PDE, reaction-diffusion-advection equation, regularization, convergence.

1 Introduction

In this paper, we develop an asymptotic expansion-regularization method for two-dimensional inverse source problems which arise from time-dependent singularly perturbed partial differential equations (PDEs). To illustrate our ideas, we take the following inverse problem as an example:

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(IP): Given noisy data $\{u^\delta(x,y,t_0), u_x^\delta(x,y,t_0), u_y^\delta(x,y,t_0)\}$ of $\{u(x,y,t), u_x(x,y,t), u_y(x,y,t)\}$ at the $n\cdot m$ location points $\{x_i,y_j\}_{i,j=0}^{n,m}$ and at the time point t_0, find the source function $f(x,y)$ such that (u,f) satisfies the dimensionless nonlinear autowave model

$$\begin{cases} \mu\Delta u - \dfrac{\partial u}{\partial t} = -u\left(k\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\right) + f(x,y), & x\in\mathbb{R}, \quad y\in(-a,a), \quad t\in(0,T], \\ u(x,y,t,\mu) = u(x+L,y,t,\mu), & x\in\mathbb{R}, \quad y\in[-a,a], \quad t\in[0,T]\equiv\bar{\mathcal{T}}, \\ u(x,-a,t,\mu) = u^{-a}(x), \quad u(x,a,t,\mu) = u^a(x), & x\in\mathbb{R}, \quad t\in[0,T], \\ u(x,y,0,\mu) = u_{init}(x,y,\mu), & x\in\mathbb{R}, \quad y\in[-a,a], \end{cases} \tag{1.1}$$

where $u(x,y,t)$ represents the temperature or oil saturation, $\mu \ll 1$ denotes kinematic viscosity, the positive constant k is the medium anisotropy coefficient, and $f(x,y)$ is the source function. We assume that the function $f(x,y)$ is L-periodic in the variable x and sufficiently smooth in the region $(x,y): \mathbb{R}\times\bar{\Omega}\,(\Omega\equiv(-a,a))$, that the functions $u^{-a}(x)$, $u^a(x)$ are L-periodic and sufficiently smooth in $x\in\mathbb{R}$, and that $u_{init}(x,y,\mu)$ is a sufficiently smooth function in $(x,y):\mathbb{R}\times\Omega$ and L-periodic in x, satisfying

$$u_{init}(x,-a,\mu) = u^{-a}(x), \quad u_{init}(x,a,\mu) = u^a(x).$$

In this paper, we focus on the speed, location, and width of the border between two regions – the region with a small dimensionless value u and the region with its high value. The domain of the function describing the moving front contains a subdomain in which the function has a large gradient. Interest in front-type solutions is associated with combustion problems [23] or nonlinear acoustic waves [37].

Note that the inverse source problem (IP) is ill-posed (see [19]), we should therefore employ the regularization methods to obtain a meaningful approximate source function. Within the framework of Tikhonov regularization, the (IP) can be converted to the following PDE-constrained optimization problem:

$$\begin{aligned} \min_f \sum_{i=0}^{n}\sum_{j=0}^{m} \Bigg\{ & \left[u(x_i,y_j,t_0) - u^\delta(x_i,y_j,t_0)\right]^2 + \left[\frac{\partial u}{\partial x}(x_i,y_j,t_0) - \frac{\partial u^\delta}{\partial x}(x_i,y_j,t_0)\right]^2 \\ & + \left[\frac{\partial u}{\partial y}(x_i,y_j,t_0) - \frac{\partial u^\delta}{\partial y}(x_i,y_j,t_0)\right]^2 \Bigg\} + \varepsilon\mathcal{R}(f), \end{aligned} \tag{1.2}$$

where u solves the nonlinear PDE (1.1) with a given f, $\mathcal{R}(f)$ denotes the regularization term, and $\varepsilon > 0$ is the regularization parameter.

Although the conventional formulation (1.2) for (IP) is straightforward, the numerical realization is difficult in many applications for the following three reasons:

(1) The regularization term $\mathcal{R}(f)$ reflects the a priori information about the source function, which is hard to obtain in practice.