CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2023-0001

Vol. **4**, No. 4, pp. 696-720
December 2023

A Priori Bounds for Elastic Scattering by Deterministic and Random Unbounded Rough Surfaces

Tianjiao Wang[1], Yiwen Lin[2] and Xiang Xu[1,*]

[1] *School of Mathematical Sciences, Zhejiang University, Hangzhou 310058, P.R. China.*
[2] *School of Mathematical Sciences and Institute of Natural Sciences, Shanghai Jiao Tong University, Shanghai 200240, P.R. China.*

Received 4 January 2023; Accepted 4 May 2023

Abstract. This paper investigates the elastic scattering by unbounded deterministic and random rough surfaces, both of which are assumed to be graphs of Lipschitz continuous functions. For the deterministic case, an a priori bound explicitly dependent on frequencies is derived by the variational approach. For the scattering by random rough surfaces with a random source, well-posedness of the corresponding variation problem is proved. Moreover, a similar bound with explicit dependence on frequencies for the random case is also established based upon the deterministic result, Pettis measurability theorem and Bochner's integrability theorem.

AMS subject classifications: 35P25, 35R60

Key words: Elastic wave scattering, unbounded rough surface, variation problem, a priori bound.

---

1 Introduction

This paper considers mathematical analysis of time-harmonic elastic waves scattered by unbounded deterministic and random rough surfaces in two dimensions. Elastic scattering problems have received intensive attentions both in mathematics and engineering because of their wide-ranging applications in seismology and geophysics (see [1,16,21]). Mathematically, elastic wave scattering can be formulated as a boundary value problem of the Navier equation which is more complicated than electromagnetic and acoustic equations.

Considerable efforts have been devoted to electromagnetic and acoustic rough surface scattering. For instance, Chandler-Wilde and Zhang [11] proposed an upward radiation condition (UPRC) of the Helmholtz equation and studied the Green function and

*Corresponding author. *Email address:* `xxu@zju.edu.cn` (X. Xu)

potentials of electromagnetic scattering by rough surfaces. Furthermore, they employed an integral equation method to prove the corresponding existence and uniqueness in [12]. Moreover, variation approaches are utilized to prove the well-posedness based on Rellich identities which imply an a priori bound with explicit dependence to the wave number in [10]. Recently, Chandler-Wilde and Elschner [9] extended the well-posedness in weighted Sobolev spaces by variation approaches and used the finite element method with perfectly matched layer technique to solve acoustic scattering by rough surfaces. For the scattering with tapered incident wave by fractal rough surface, Zhang *et al.* [24] used regularized conjugate gradient method to reconstruct the surface. Zhang *et al.* [25] obtained the Fréchet derivative of the scattered field which can be used to give numerical methods for shape reconstruction from multi-angle and multi-frequency data. Similar results for general unbound rough surface were given by Zhang and Ma in [23]. Bao and Zhang realized the reconstruction from multi-frequency phaseless data in [7] and obtained the uniqueness and existence for direct problem and uniqueness for inverse problem based on boundary integral equations in [8]. Numerical method for recovering localized perturbation of unbounded surface via near-field was proposed by Bao and Lin [5].

Compared to electromagnetic and acoustic scattering, results on elastic scattering from unbounded rough surfaces are relatively fewer. Arens [2–4] investigated the Green tensor, elastic potentials, UPRC and proved uniqueness and existence by integral equation methods. Elschner and Hu [14] deduced a transparent boundary condition and proved existence and uniqueness by variation approaches based on the Rellich identities. Furthermore, they studied solvability in weighted Sobolev spaces, on which they based to prove the existence and uniqueness of elastic scattering by unbounded rough surfaces with a plane or point source incident wave in [15]. Recently Hu *et al.* [18] generalized the similar results for three-dimensions.

For random cases, Warnick and Chew [22] proposed a numerical method to solve electromagnetic scattering from random rough surfaces. Pembery and Spence [20] considered the Helmholtz equation in random media and proposed a general framework to study the variation problem, which overcomes the difficulties on both lacks of coercivity and the necessary compactness in Bochner's spaces. Bao *et al.* [6] extended this general framework to obtain an explicit stability result with respect to the wave number for electromagnetic scattering by random periodic surfaces.

In this paper, we derive an a priori bound explicitly dependent on the frequencies and the truncated height for the deterministic elastic scattering by rough surfaces based on Rellich identities. Different from electromagnetic scattering, direct applying Rellich identities is not enough for elastic scattering. By the method in [14], we use the a priori bound for Helmholtz equations and construct a boundary value problem of a Helmholtz equation to overcome the difficulty. Moreover, for the random case, we prove the well-posedness of the stochastic variation problem and extend the explicit bound based on the framework in [20]. The main difference with [20] is that the variation forms for different samples are defined in different Banach spaces. So we need to use the method of