East Asian Journal on Applied Mathematics Vol. **13**, No. 4, pp. 791-812
doi: 10.4208/eajam.2021-275.210722 November 2023

A Mixed-Type Circulant Preconditioner for a Nonlocal Elastic Model

Yuan-Yuan Huang and Siu-Long Lei*

Department of Mathematics, University of Macau, Macau, China.

Received 16 September 2021; Accepted (in revised version) 21 July 2022.

Abstract. A linear system from finite difference discretization of a generalized nonlocal elastic model was studied, where the model is composed of a Riesz potential operator with a fractional differential operator. Some properties of the coefficient matrix are proven theoretically and it is found that the linear system is very ill-conditioned when the parameter in the long-range hydrodynamic interactions is close to zero. Therefore, the usual Krylov subspace method with the Strang-Strang circulant preconditioner loses the power of preconditioning so that the iterative method converges slowly. Here the problem is fixed by utilizing a mixed-type circulant preconditioner obtained by both Strang's and Chan's circulant approximations. The invertibility of the preconditioner and a small-norm-low-rank decomposition of the difference matrix of the coefficient matrix and the preconditioner are shown theoretically under certain conditions. Numerical examples are given to illustrate the efficiency of the proposed fast solver.

AMS subject classifications: 65M10, 78A48

Key words: Generalized nonlocal elastic model, peridynamic, fractional differential operator, Toeplitz linear system, circulant preconditioner.

1. Introduction

Continuum elastic models have been widely used to study the dynamics of real physical systems in such applications as flexible polymers, growing interfaces, and membranes [7, 17, 29]. Recently, a generalized elastic model was developed in [25] to study a generalization and relations of these elastic models. This model is expressed as a composition of a Riesz-like potential operator with a fractional differential operator [1, 23]. In a sense, the generalized elastic model can be viewed as a composition of a peridynamic model with a fractional differential equation model. Due to the nonlocal nature and complexity of these nonlocal models, the corresponding numerical methods typically generate dense or full coefficient matrices, which require $\mathcal{O}(N^3)$ computational complexity and $\mathcal{O}(N^2)$ storage by direct methods, where N signifies the number of grid points. Extensive research

*Corresponding author. *Email addresses:* `yc07487@um.edu.mo` (Y.-Y. Huang), `sllei@um.edu.mo` (S.-L. Lei)

has been conducted on the development of fast and accurate numerical methods for nonlocal models [6, 9, 26]. In [8], the authors decomposed the nonlocal elastic model as two systems and used the Meerschaert-Tadjeran finite difference and a collocation method to discretize them respectively. Strang's preconditioner was proposed to solve the linear system by preconditioned fast Krylov subspace method. However, they did not give theory analysis. In this paper, we study the linear system from the discretization of the generalized elastic model developed in [8]. The coefficient matrix of the linear system can be written as a product of two matrices AB where the dimensions of A and B are $N \times (N+2)$ and $(N+2) \times N$, respectively. As pointed out in [8], matrix A is related to the potential operator and its central square part $\tilde{A}$ is a symmetric Toeplitz matrix. The matrix B is related to the fractional differential operator and its central square part $\tilde{B}$ is generally nonsymmetric and possesses a Toeplitz-like structure. Here we further prove that the square matrix $\tilde{A}$ is positive definite by making use of an appropriate congruent transformation. Besides, it is noticed that $AB = \tilde{A}\tilde{B} + R$, where R is a matrix with rank being equal to 2. When the diffusivity coefficients are positive constant, it is shown that the real part of every eigenvalue of $\tilde{A}\tilde{B}$ is negative. Besides, all eigenvalues of R are real and nonpositive under certain conditions. Hence, it is conjectured that the spectrum of the coefficient matrix AB is included in the set $\{z \in \mathbb{C} \,|\, \mathrm{Re}(z) < 0\}$, where $\mathbb{C}$ is the set of all complex numbers and $\mathrm{Re}(z)$ denotes the real part of z. Besides the properties of AB, an effective preconditioner is proposed for solving linear systems with coefficient matrix AB. By approximating $\tilde{A}$ and $\tilde{B}$ by using Chan's circulant matrix [5, 13, 19] and Strang's circulant matrix [3, 4, 13] respectively, a mixed-type circulant preconditioner is obtained. Since circulant matrices can be diagonalized by Fourier matrix, the proposed preconditioner can be inverted efficiently via fast Fourier transform (FFT) [8]. It is also found that the matrix AB is very ill-conditioned when the parameter in the long-range hydrodynamic interactions is close to zero. The proposed mixed-type circulant preconditioner performs very well in the ill-conditioned case while the preconditioner proposed in [8] fails to reduce the number of iterations in the preconditioned GMRES method. Besides, we note that band-Toeplitz preconditioners can be also a good choice for ill-conditioned Toeplitz systems [2, 11, 12, 16, 20, 21, 24].

The paper is organized as follows. In Section 2, the elastic model and the discretized linear system are reviewed. Some properties of the coefficient matrix are given in Section 3 and a well-defined mixed-type preconditioner is proposed in Section 4. In Section 5, numerical results are reported to demonstrate the efficiency of the proposed preconditioner. Conclusions are drawn in Section 6.

2. Numerical Scheme for Model Problem

We consider the following nonconventional Dirichlet boundary-value problem of the generalized fractional elastic model:

$$\begin{aligned}&\int_0^1 \frac{1}{|x-y|^\alpha}\left(d^+(y)\frac{\partial^\beta u(y)}{\partial_+ y^\beta} + d^-(y)\frac{\partial^\beta u(y)}{\partial_- y^\beta}\right)dy \\ &= f(x), \quad x \in (0,1), \quad 0 < \alpha < 1,\end{aligned} \tag{2.1a}$$