A FINITE VOLUME ELEMENT SOLUTION BASED ON POSTPROCESSING TECHNIQUE OVER ARBITRARY CONVEX POLYGONAL MESHES

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Abstract. A special finite volume element method based on postprocessing technique is proposed to solve the anisotropic diffusion problem on arbitrary convex polygonal meshes. The shape function of polygonal finite element method is constructed by Wachspress generalized barycentric coordinate, and by adding some element-wise bubble functions to the finite element solution, we get a new finite volume element solution that satisfies the local conservation law on a certain dual mesh. The postprocessing algorithm only needs to solve a local linear algebraic system on each primary cell, so that it is easy to implement. More interesting is that, a general construction of the bubble functions is introduced on each polygonal cell, which enables us to prove the existence and uniqueness of the post-processed solution on arbitrary convex polygonal meshes with full anisotropic diffusion tensor. The optimal H^1 and L^2 error estimates of the post-processed solution are also obtained. Finally, the local conservation property and convergence of the new polygonal finite volume element solution are verified by numerical experiments.

Key words. Finite volume element solution, postprocessing technique, convex polygonal meshes, existence and uniqueness, H^1 and L^2 error estimates.

1. Introduction

Finite volume method (FVM) is a popular and practical numerical method for solving partial differential equations, and it is widely used in computational fluid dynamics, computational heat transfer and other fields. The local conservation is an important property of FVM, and it is desirable in multiphase flow in porous media, energy conservation in thermodynamics and many other problems. Finite volume element method (FVEM) is usually regarded as a special type of FVM, where the solution space is the same as the classical finite element method. The mathematical development of FVEM can be found in [18, 20, 37]. For the two dimensional diffusion problems, most existing works of FVEM are only concentrated on triangular meshes (e.g. [1, 5, 6, 7, 9, 34, 36, 39]) or quadrilateral meshes (e.g. [14, 15, 19, 21, 23, 28, 38]). Polygonal meshes offer greater flexibility in mesh generation, merging and refinement, and they have been applied in many fields, such as computational fluid dynamics, topology optimization, analysis of fractured materials and crack propagation and so on. Thus, the construction of FVEM on polygonal meshes is an interesting and important research topic. Recently, [42] proposed a finite volume element method to solve the anisotropic diffusion equation on general convex polygonal meshes, and under the coercivity assumption, the authors proved the optimal H^1 error estimate. To our knowledge, the theoretical analysis of FVEM on arbitrary convex polygonal meshes still lags far behind. For instance, even though for the classical isoparametric bilinear FVEM, the corresponding coercivity result and optimal L^2 error analysis have not been established on arbitrary trapezoidal meshes.

As we all know, since the bilinear form of classical finite element method (FEM) is symmetry, the coercivity result can be easily obtained (e.g. [2, 3]). Once the optimal interpolation error estimate is established, the optimal error analysis (e.g. H^1 and L^2) of FEM can also be proved by some standard techniques (e.g. Aubin-Nitsche). In recent decades, based on various generalized barycentric coordinates, some researchers extend the classical FEM to polygonal meshes, where the generalized barycentric coordinates are studied in [10, 11, 12, 16, 25, 33] for incomplete references. In [30], the polygonal FEMs based on Wachspress, mean value or Laplace generalized barycentric coordinates were developed. For more studies and applications about polygonal finite element method, the readers are referred to [24, 26, 29, 32, 35] and so on. At the same time, [13] (resp. [27]) studied the interpolation error estimates of triangulation, harmonic, Wachspress and Sibson (resp. mean value) coordinates, which is crucial to the optimal error estimates in polygonal FEM and FVEM.

Regrettably, the aforementioned polygonal FEM doesn't satisfy the local conservation property in general. Thus, some researchers try to postprocess the FEM solution to obtain a new FVEM solution with the local conservation property. By postprocessing the continuous Galerkin finite element solution, [43] presented a high order finite volume element solution for the elliptic problem on triangular and quasi-parallelogram meshes, which has the local conservation property and preserves the H^1 and L^2 error estimates. Later, [40] generalized the theoretical results in [43] to the anisotropic diffusion equation on arbitrary trapezoidal meshes. Recently, by introducing some new bubble functions, [41] improved the postprocess technique in [43, 40], such that the new theoretical findings cover arbitrary triangular and convex quadrilateral meshes for the anisotropic diffusion equation with any full diffusion tensor. In addition, there are many other research results for the local conservative method based on FEM, e.g. [4, 8, 17, 22, 31].

Compared with the previous works, this article has several contributions. Firstly, by introducing a unified construction of bubble functions, we establish the existence and uniqueness, optimal H^1 and L^2 error estimates of the post-processed solution for the anisotropic diffusion equation on arbitrary convex polygonal meshes. Secondly, we note that the coercivity result in [42] does not cover general convex polygonal meshes, and the L^2 error estimate of it has not been established. Different from [42], here we present another routine to obtain a new polygonal finite volume element solution for solving the anisotropic diffusion equation, and the stability and convergence are verified on arbitrary convex polygonal meshes.

The rest of this article is organized as follows. In Section 2, we define some notations and introduce the polygonal finite element method based on Wachspress generalized barycentric coordinate. By postprocessing the polygonal finite element solution, in Section 3 we reach a polygonal finite volume element solution, and the local conservation, existence and uniqueness of the post-processed solution are verified in Section 4. The optimal error estimates in H^1 and L^2 norms are proved in Section 5. In Section 6, we present some numerical results to validate the accuracy, local conservation property and mesh flexibility of the proposed method. Some conclusions are given in the last section.

2. A polygonal finite element method

Consider the following anisotropic diffusion problem

$$-\nabla \cdot (\Lambda \nabla u) = f \quad \text{in } \Omega, \tag{1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{2}$$