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# Some Summation Theorems for Truncated Clausen Series and Applications

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**Abstract.** The main aim of this paper is to derive some new summation theorems for terminating and truncated Clausen's hypergeometric series with unit argument, when one numerator parameter and one denominator parameter are negative integers. Further, using our truncated summation theorems, we obtain the Mellin transforms of the product of exponential function and Goursat's truncated hypergeometric function.

**Key Words**: Watson summation theorem, Whipple summation theorem, Dixon summation theorem, Saalschütz summation theorem, Truncated series, Hypergeometric summation theorems, Mellin transforms.

**AMS Subject Classifications**: 33C05, 33C20, 44A10

## 1 Introduction

It should be noted here that whenever hypergeometric and generalized hypergeometric functions can be summed to be expressed in terms of Gamma functions, the results are very important from a theoretical and an applicable point of view. Until 1990, only few classical summation theorems such as of Gauss, Gauss's second, Kummer, and Bailey for the series ${}_2F_1$, and Watson, Dixon, and Whipple for the series ${}_3F_2$ were known. They play an important role in the theory of generalized hypergeometric series. Subsequently, some progress has been made in generalizing these classical summation theorems (see [6–9,11, 14,15]).

## 2 Preliminaries results

The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

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The Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ (see [16, p. 22, Eq. (1), p. 32, Eqs. (8) and (9)], see also [18, p. 23, Eqs. (22) and (23)]), is defined by

$$\begin{aligned}(\alpha)_p :=&\frac{\Gamma(\alpha+p)}{\Gamma(\alpha)}\\ =&\begin{cases}1, & (p=0;\ \alpha\in\mathbb{C}\setminus\{0\}),\\ \alpha(\alpha+1)\cdots(\alpha+n-1), & (p=n\in\mathbb{N};\ \alpha\in\mathbb{C}\setminus\mathbb{Z}_0^-),\\ \dfrac{(-1)^n k!}{(k-n)!}, & (\alpha=-k,\ p=n,\ n,k\in\mathbb{N}_0,\ 0\le n\le k),\\ 0, & (\alpha=-k,\ p=n,\ n,k\in\mathbb{N}_0;\ n>k),\\ \dfrac{(-1)^n}{(1-\alpha)_n}, & (p=-n;\ n\in\mathbb{N};\ \alpha\in\mathbb{C}\setminus\mathbb{Z}),\end{cases}\end{aligned} \tag{2.1}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

The generalized hypergeometric function ${}_pF_q$ (see [16, pp. 73-74], see also [1]), is defined by

$${}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\cdots,\alpha_p;\\ \\ \beta_1,\beta_2,\ldots,\beta_q;\end{array}\ z\right] = {}_pF_q\left[\begin{array}{c}(\alpha_p);\\ \\ (\beta_q);\end{array}\ z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}. \tag{2.2}$$

By convention, a product over the empty set is unity, $\big(p,q \in \mathbb{N}_0;\ p \leqq q+1\ ;\ p \leqq q$ and $|z| < \infty;\ p = q+1$ and $|z| < 1;\ p = q+1, |z| = 1$ and $\Re(\omega) > 0;\ p = q+1, |z| = 1, z \neq 1$ and $-1 < \Re(\omega) \leq 0\big)$, where

$$\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j, \quad \big(\alpha_j \in \mathbb{C}\ (j=1,2,\ldots,p);\quad \beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-\,(j=1,2,\ldots,q)\big),$$

where $\Re$ denotes the real part of complex number throughout the paper.

A finite series identity (reversal of the order of terms in finite summation) is given by

$$\sum_{n=0}^{m}\Phi(n) = \sum_{n=0}^{m}\Phi(m-n), \quad m \in \mathbb{N}_0. \tag{2.3}$$

The truncated hypergeometric series is given by: the sum of the first $(m+1)$-terms of