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Well-Posedness for Fractional (p, q)-Difference Equations Initial Value Problem*

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Abstract In this paper, we investigate a class of the fractional (p, q)-difference initial value problem with the fractional (p, q)-integral boundary conditions with the aid of the method of successive approximations(Picard method) and fractional (p, q)-Gronwall inequality, obtaining sufficient conditions for the existence, uniqueness and continuous dependence results of solutions.

Keywords Well-posedness, fractional (p, q)-difference equation, initial value problem, fractional (p, q)-Gronwall inequality

MSC(2010) 26A33, 34A12, 39A70.

1. Introduction

Fractional calculus is an interesting and old subject that is a generalization of ordinary differentiation and integration. The study of the q-difference equation appeared at the beginning of the 21st century and was initially developed by Jackson [8]and Carmichael [3]. So far, q-difference equations have been infiltrated into various subjects (see [1,6,7]). Some basic definitions and properties of q-difference calculus can be found in [9]. In addition, fractional calculus has proved to be a valuable tool in many fields of science and engineering such as control, fluid flow, mechanics and electrical networks (see the papers [2,17–19] and the references therein).

In recent years, with the development of science and technology and the advancement of fractional order theory, many researchers have developed the theory of quantum calculus based on the two parameters p and q. The fractional-order (p, q)-difference equations have been widely used in physical sciences, Lie groups, special functions, hypergeometric series, Bezier curves and approximations. Some basic results of (p, q)-difference calculus can be found in [4, 10, 11, 13, 20, 21].

In particular, there are few works that have considered the fractional (p, q)-difference equations. In 2020, Soontharanonl and Sitthiwirattham [22] studied the existence of a class of fractional (p, q)-difference equation. A class of fractional (p, q)-integrodifference equation with periodic fractional (p, q)-integral boundary conditions was considered with Banach and Schauder's fixed point theorems [25]. In 2021, Qin and Sun [15] proved the existence and uniqueness of positive solutions for fractional (p, q)-difference equation using some standard fixed point theorems.

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In this year, Qin and Sun [16] studied the solvability and stability for a class of singular fractional (p,q)-difference equation using Arzela's lemma and fractional (p,q)-Gronwall inequality. In the same year, the boundary value problem of a class of fractional (p,q)-difference Schrödinger equations was studied by Qin and Sun [14]. In 2022, Neang, Nonlaopon and Tariboon [12] investigated the separate local boundary value conditions of fractional (p,q)-difference equation and obtained the existence and uniqueness of solutions based on some standard fixed point theorems. For some developments concerning the existence(and uniqueness) in fractional (p,q)-difference equations, we refer to [5, 23, 24] and references therein.

Inspired by the works mentioned above, we investigate the existence, uniqueness and continuous dependence of the solution to the fractional (p,q)-difference initial value problem(IVP)

$$\begin{cases} D^{\alpha}_{p,q}u(t) = f(p^{\alpha}t, u(p^{\alpha}t)), & t \in (0,1], \\ I^{1-\alpha}_{p,q}u(t)|_{t=0} = \eta, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $0 < q < p \leq 1$, and $D^{\alpha}_{p,q}$ is an Riemann-Liouville type fractional (p,q)-difference operator.

Few papers have investigated the fractional (p,q)-difference equations, since fractional (p,q)-operator was defined lately. Compared with the papers [12, 14–16] and [5, 22–25], the main novelty of this paper is as follows. We apply the method of successive approximations(Picard method) and fractional (p,q)-Gronwall inequality to study the existence, uniqueness and continuous dependence results of the solution for problem (1.1), which is the first and probably the only work in this direction. To the author's knowledge, there is no result on the continuous dependence of the solution for fractional (p,q)-difference equation. In this paper, we aim to fill this margin to some extent. Thus, our works are new and meaningful.

The organization of our paper is as follows. In Section 2, we present some basic definitions and preliminaries results. In Section 3, we prove the main results of this paper, which include the existence, uniqueness and continuous dependence of the solution to problem (1.1).

2. Preliminaries

In this section, we present some concepts of fractional (p,q)-difference calculus and some necessary basic preliminaries. Let $0 < q < p \leq 1$. Define

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & k \in \mathbb{N}, \\ 1, & k = 0, \end{cases}$$

$$[k]_{p,q} := \begin{cases} \dfrac{p^k - q^k}{p-q} = p^{k-1}[k]_{\frac{q}{p}}, & k \in \mathbb{N}, \\ 0, & k = 0, \end{cases}$$