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Dynamics of a Degenerately Damped Stochastic Lorenz-Stenflo System*

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Abstract It seems that little has been known about the sensitivity of steady states in stochastic systems. This paper proves the conditions for the existence of an invariant measure in a degenerately damped stochastic Lorenz-Stenflo model. Precisely, the solution is proved to be a nice diffusion via the Lie bracket technique and non-trivial Lyapunov functions. The finiteness of the expected positive recurrence time entails the existence problem. On the other hand, a cut-off function is constructed to show the non-existence result through a proof by contradiction. For other interesting cases, the expected recurrence time is shown to be infinite.

Keywords Lorenz-Stenflo system, invariant measure, Lyapunov function, noise-induced stabilization.

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1. Introduction

To describe the low-frequency and short-wavelength acoustic-gravity perturbations in the atmosphere, Stenflo [22] derived a four-dimensional continuous-time dynamical system given by

$$\begin{cases} \dfrac{dx}{dt} = a(y - x) + rw, \\ \dfrac{dy}{dt} = cx - y - xz, \\ \dfrac{dz}{dt} = xy - bz, \\ \dfrac{dw}{dt} = -x - aw, \end{cases} \tag{1.1}$$

where x, y, z, w are state variables of the so-called Lorenz-Stenflo equation (1.1), and positive parameters a, c, r are the Prandtl, generalized Rayleigh and rotation numbers respectively, and b is the geometric parameter.

Obviously, one can reduce (1.1) to the usual Lorenz system in [15] with interesting mathematical properties, if the rotation of the earth is not considered. In

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the past decades, many scholars studied its complex dynamical behaviors such as boundedness [24, 33], periodicity [17, 28], bifurcation [26, 29, 30, 34], synchronization [6], chaotic and hyperchaotic dynamics [9, 19, 25, 27] as well as the influence of Lévy noise [12].

Notice that the geometric parameter b is strictly positive as shown in the derivation of (1.1). But it will tend to zero under the sufficiently large generalized Rayleigh number. On the other hand, the so-called Homogeneous Rayleigh-Bénard (HRB) system was established with $b \le 0$ appearing in the temperature equation [3, 4]. Indeed, a similar degeneracy effect was observed in a certain zero Prandtl limit for modeling mantle convection [20, 23]. Therefore, it is natural to investigate the corresponding dynamics of (1.1), when $b \le 0$. However, it is straightforward that the corresponding solution to (1.1) on the z-direction explodes in finite time under the initial conditions ($x_0 = y_0 = w_0 = 0, z_0 \neq 0$) provided that $b < 0$. As for the case $b = 0$, any point on the z-axis becomes an equilibrium. Thus, one can prove the existence of singularly degenerate heteroclinic cycles, even there is no compact global attractor in this situation. Therefore, both embarrassing cases motivate us to study the possibility of stabilizing the dynamics by adding external noise perturbations.

It is well-known that arbitrary small additive noise can stabilize an explosive ordinary differential equation (ODE) [16, 18]. If, in addition, the corresponding Markov process admits an invariant probability measure, it corresponds to the so-called noise-induced stabilization problem. In this respect, considerable interest has already been shown in studying stationary states, stable oscillations and the related work [1, 2, 5, 7, 8, 10, 11, 13, 14, 21, 31, 32].

Motivated by the aforementioned discussion, we are interested in the stochastic Lorenz-Stenflo system

$$\begin{cases} dx = (a(y-x) + rw)dt + \sqrt{2\kappa_1}dB_1, \\ dy = (cx - y - xz)dt + \sqrt{2\kappa_2}dB_2, \\ dz = (xy - bz)dt + \sqrt{2\kappa_3}dB_3, \\ dw = (-x - aw)dt + \sqrt{2\kappa_4}dB_4, \end{cases} \tag{1.2}$$

where $B_i, i = 1, 2, 3, 4$ are independent and standard Brownian motions, and $\kappa_i \ge 0, i = 1, 2, 3, 4$ represent the intensity of random noise and other parameters conform to the ones in system (1.1). To ensure system (1.2) is genuinely stochastic, we require that at least one κ_i is positive.

In the absence of noise, we know that the solutions are explosive or have no compact global attractor when $b \le 0$. The interesting question here is whether the presence of noise induces the existence and the number of invariant probability measure for the generated Markov transition semigroup.

The paper aims to solve the noise-induced stabilization problem of (1.2) with additive Brownian noise by applying the way in [7, 32]. More precisely, we first state the philosophy of proving the existence of a unique invariant probability measure for the Markov transition semigroup generated by (1.2). The first step is to verify the non-explosion of the solution to (1.2) under a suitable Lyapunov function and Young's inequality. Then, Lie bracket over the vector fields shows that such a solution is a nice diffusion. The final step is to acquire the globally finite expected returns to some compact set by constructing another Lyapunov function. As for the non-existence under a highly degenerate noise, we construct a cut-off function and