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Existence and Non-Existence of Positive Solutions for a Discrete Fractional Boundary Value Problem

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Abstract In this work, we deal with two-point boundary problem for a finite nabla fractional difference equation. First, we establish an associated Green's function and state some of its properties. Under suitable conditions, we deduce the existence and non-existence of positive solutions to the considered problem. Finally, we construct a few examples to illustrate the established results.

Keywords Nabla fractional difference, dirichlet boundary conditions, boundary value problem, Green's function, fixed-point, positive solution, eigenvalue

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1. Introduction

In 1695, L'Hospital inquired Leibniz on the differential operator $\dfrac{d^n}{dt^n}$, "what if the order is $\frac{1}{2}$", to which Leibniz replied, "it will lead to a paradox from which one-day useful consequences will be drawn". This question gave birth to a branch of mathematics that we know today as the fractional calculus [8, 30]. Although it almost started at the same time as differential calculus, most of the early developments of fractional calculus were confined to the basement for a long time. Today, fractional calculus has been successfully applied in mathematical modelling for medical sciences, computational biology, economics, physics and several areas of engineering. For further applications and historical literature, we refer to a few classical texts on fractional calculus here by Miller and Ross [26], Samko, Kilbas and Mariche [29], Podlubny [28] and Kilbas, Srivastava and Trujillo [23].

On the other hand, discrete fractional calculus deals with arbitrary order differences and sums defined on a discrete domain with a forward (delta) or a backward (nabla) operator. The theory of discrete fractional calculus is relatively new with the most notable works done in the past decade. The notion of fractional difference and sum can be traced back to the work of Gray and Zhang [13] as well as Miller and Ross [27]. In this line, Atici and Eloe [16] developed nabla fractional Riemann–Liouville difference operator, initiated the study of nabla fractional initial value problem and established the exponential law, product rule and nabla Laplace transform. Following their works, the contributions of several mathematicians have made the theory of discrete fractional calculus a fruitful field of research in science

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and engineering. Here, we refer to a recent monograph by Goodrich and Peterson [11] and the references therein, which is an excellent source for all those who wish to work in this field.

The study of boundary value problems (BVPs) has a rich historical background and can be traced back to the work of Euler and Taylor on vibrating strings. On the discrete fractional side, there is a sudden growth of the interest in the development of nabla fractional BVPs. Many authors have studied nabla fractional BVPs recently. To name a few, Ahrendt [2], Goar [10] and Ikram [16] worked with self-adjoint Caputo nabla BVPs. Brackins [7] studied a particular class of self-adjoint Riemann–Liouville nabla BVPs and derived the Green's function associated with it along with a few of its properties. Gholami and Ghanbari [9] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla BVP with Dirichlet boundary conditions. Jonnalagadda [17–22] analysed some qualitative properties to two-point non-linear Riemann–Liouville nabla BVPs associated with a variety of boundary conditions.

Our purpose of this article is to establish sufficient conditions on the existence and non-existence of positive solutions to the following two-point non-linear nabla fractional BVP with parameter $\beta > 0$, using Guo–Krasnoselskii fixed-point theorem [1].

$$\begin{cases} -\left(\nabla^{\alpha}_{\rho(a)} u\right)(t) = \beta f(t, u), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_3$, $1 < \alpha < 2$ and $f : \mathbb{N}^b_a \times \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$.

The present article is organized as follows. Section 2 contains a few preliminaries on nabla fractional calculus. In Sections 3 and 4, we present the main results on the existence and non-existence of positive solutions to (1.1). Finally, we conclude this article with a few examples to demonstrate the applicability of our main results.

2. Preliminaries

Denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{Z}^+$ respectively. We use the following notations, definitions and known results of nabla fractional calculus [11]. Assume that empty sums and products are 0 and 1 respectively.

Definition 2.1. For $a \in \mathbb{R}$, the sets $\mathbb{N}_a$ and $\mathbb{N}^b_a$, where $b - a \in \mathbb{Z}^+$, are defined by

$$\mathbb{N}_a = \{a, a+1, a+2, ...\}, \quad \mathbb{N}^b_a = \{a, a+1, a+2, ..., b\}.$$

Definition 2.2. We define the backward jump operator, $\rho : \mathbb{N}_{a+1} \longrightarrow \mathbb{N}_a$, by

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{a+1}.$$

Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by $\big(\nabla u\big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$, and the N^{th}-order nabla difference of u is defined recursively by $\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t)$, for $t \in \mathbb{N}_{a+N}$.

Definition 2.3. [11] Let $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$. The generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t + r)}{\Gamma(t)}.$$