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Dynamics of Stochastic Ginzburg-Landau Equations Driven by Colored Noise on Thin Domains*

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Abstract This work is concerned with the asymptotic behaviors of solutions to a class of non-autonomous stochastic Ginzburg-Landau equations driven by colored noise and deterministic non-autonomous terms defined on thin domains. The existence and uniqueness of tempered pullback random attractors are proved for the stochastic Ginzburg-Landau systems defined on $(n+1)$-dimensional narrow domain. Furthermore, the upper semicontinuity of these attractors is established, when a family of $(n+1)$-dimensional thin domains collapse onto an n-dimensional domain.

Keywords Stochastic Ginzburg-Landau equation, colored noise, thin domain, random attractor, upper semicontinuity

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1. Introduction

In this paper, we investigate the asymptotic behavior of solutions of the following non-autonomous stochastic Ginzburg-Landau equations driven by colored noise on $\mathcal{O}_\varepsilon$ with Neumann boundary conditions: for $t > \tau$ with $\tau \in \mathbb{R}$ and $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$,

$$\begin{cases} \dfrac{\partial \hat{u}^\varepsilon}{\partial t} - (1+\mathrm{i}\mu)\Delta \hat{u}^\varepsilon + \rho \hat{u}^\varepsilon = f(t,x,\hat{u}^\varepsilon) + G(t,x) + R(t,x,\hat{u}^\varepsilon)\zeta_\delta(\theta_t\omega), \quad x \in \mathcal{O}_\varepsilon, \\ \dfrac{\partial \hat{u}^\varepsilon}{\partial \nu_\varepsilon} = 0, \quad x \in \partial\mathcal{O}_\varepsilon, \end{cases} \tag{1.1}$$

with the initial condition

$$\hat{u}^\varepsilon(\tau, x) = \hat{u}^\varepsilon_\tau(x), \quad x \in \mathcal{O}_\varepsilon, \tag{1.2}$$

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where $\hat{u}^{\varepsilon}(t,x)$ is a complex-valued function on $\mathbb{R}\times\mathcal{O}_{\varepsilon}$. In (1.1), i is the imaginary unit, and μ,ρ are real constants and $\rho>0$. ν_{ε} is the unit outward normal vector to $\partial\mathcal{O}_{\varepsilon}$. The so-called thin domain $\mathcal{O}_{\varepsilon}$ (ε small) is given by

$$\mathcal{O}_{\varepsilon}=\Big\{x=(x^*,x_{n+1})|\ x^*=(x_1,x_2,\cdots,x_n)\in\mathcal{Q},\ 0<x_{n+1}<\varepsilon g(x^*)\Big\} \tag{1.3}$$

with $0<\varepsilon\le 1$ and $g\in C^2(\overline{\mathcal{Q}},(0,+\infty))$, where $\mathcal{Q}$ is a smooth bounded domain in $\mathbb{R}^n$. Since $g\in C^2\big(\overline{\mathcal{Q}},(0,+\infty)\big)$, there exist two positive constants β_1 and β_2 such that

$$\beta_1\le g(x^*)\le\beta_2,\quad \forall\ x^*\in\overline{\mathcal{Q}}. \tag{1.4}$$

Denote $\mathcal{O}=\mathcal{Q}\times(0,1)$ and $\widetilde{\mathcal{O}}=\mathcal{Q}\times(0,\beta_2)$ which contain $\mathcal{O}_{\varepsilon}$ for $0<\varepsilon\le 1$. The nonlinearity f and the body force G satisfy some conditions, which are to be specified later. $\zeta_{\delta}(\theta_t\omega)$ with $0<\delta\le 1$ is an Ornstein-Uhlenbeck (O-U) process (also known as a colored noise).

The O-U process is a stationary Gaussian process with zero mathematical expectation, and the O-U process is the only existing Markovian Gaussian colored noise (see, e.g. [6] and [23]). Furthermore, the O-U process is also called a colored noise, because its power spectrum is not flat compared with the white noise (see [2,7,9,23–25,28,30]).

As we know, the Wiener process W can be chosen as a stochastic process to represent the position of the Brownian particle. But the velocity of the particle cannot be obtained from the Wiener process because of the nowhere differentiability of the sample paths of W. However, the O-U process was originally constructed to approximately describe the stochastic behavior of the velocity [25,30]. Hence, it can be further used to determine the position of the particle. Furthermore, as demonstrated in [23], in many complex systems, stochastic fluctuations are actually correlated. Therefore, they should be modeled by colored noise rather than white noise.

During the study of stochastic dynamics, one of the most crucial issues arises from the modeling of random forcing. To study such a random forcing, we need to consider the time scale τ_d of the deterministic system and the time scale τ_r of the random forcing. The stochastic forcing is modeled in different ways based on the ratio of τ_r/τ_d. If $\tau_r/\tau_d\gg 1$, and the dynamical system is very slow with respect to the temporal variability of its random drivers. Hence, the random forcing could be modeled as white noise. If $\tau_r/\tau_d\simeq 1$, then the dynamics of the system is sensitive to the autocorrelation of the random forcing, and therefore the random forcing should be modeled by colored noise. Based on these considerations, the colored noise has been used in many works to study the dynamics of physical and biological system (see, e.g. [2,7,12–14,23,25,30] and the reference therein).

As $\varepsilon\to 0$, the thin domain $\mathcal{O}_{\varepsilon}$ collapses to an n-dimensional domain. In this paper, we will see that the limiting behavior of the equation is determined by the following system on the lower dimensional spatial domain $\mathcal{Q}$: for $t>\tau$ with $\tau\in\mathbb{R}$