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On the Boundary of the Attraction Basin in a Class of Piecewise Linear Systems*

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Abstract In this paper, we investigate the boundary of the attraction basin of a class of piecewise linear systems arising from anti-stable linear systems with saturated linear state feedback. In three-dimensional cases, for this class of systems, we prove that the equilibrium points other than the origin lie on the boundary of the attraction basin of the origin. This gives strong evidence that the boundary of the attraction basin is homeomorphic to a sphere. Some examples are provided to illustrate the results.

Keywords Piecewise linear system, boundary of attraction basin, index, saturated state feedback

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1. Introduction

In piecewise smooth systems, saturation nonlinearities have attracted a lot of attention, because they are ubiquitous due to the inherent physical limitations of devices. For example, control systems with saturated feedback are such a class of piecewise smooth systems which have been widely studied in the literature [5, 8, 9, 12, 13, 16, 19–21, 25, 28–31, 33, 35–37]. It is easy to see that any stabilizing linear feedback would locally stabilize a linear system with saturated actuators in the presence of actuator saturation, and the attraction basin of the origin may be bounded [3, 11, 14, 15, 17, 26, 27].

The research on the properties of the boundary of the attraction basin is of great importance because of its practical significance and a theoretical challenge from the viewpoint of dynamical system. Clearly, the structure of the boundary of the attraction basin provides much information on the dynamical behaviour of

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trajectories before approaching the origin. It is a well-known fact that a fractal boundary usually gives rise to the transient chaotic behaviour for trajectories near the boundary [1,2,18,23].

The attraction basin problem of the anti-stable linear control system with saturated state feedback is a typical problem in control systems. For a planar anti-stable linear system with saturated stabilizing state feedback, Hu, Lin and Qiu [16] showed that the boundary of the attraction basin of the closed loop system is a convex limit cycle. In view of this result, it is reasonable to conjecture that the boundary of the attraction basin in a three-dimensional anti-stable linear system with saturated stabilizing state feedback is homeomorphic to a sphere. Our numerical simulations on a lot of specific systems support this conjecture. Suppose the boundary of the attraction basin of the origin is homeomorphic to a sphere, then according to the Poincaré–Hopf theorem and its corollary [10,22], there generically exist at least two equilibrium points on the boundary of the attraction basin. Therefore, in order to verify the above conjecture, it is natural to prove that the equilibrium points (if exist) other than the origin are indeed contained in the boundary of the attraction basin as a first step.

To the best of our knowledge, no study on the boundary of the attraction basin of three-dimensional anti-stable linear systems with saturated stabilizing state feedback or even higher dimensions has appeared in the literature. Since the existence and distribution of equilibrium points affect estimation of the attraction basin, we investigate the properties of equilibrium points in such a class of n-dimensional piecewise linear systems. We prove that such an n-dimensional system has a unique equilibrium point if n is an even number, and has three equilibrium points if n is an odd number. Furthermore, for three-dimensional cases, we prove that two equilibrium points other than the origin are contained in the boundary of the attraction basin of the origin.

The paper is organized as follows. Some preliminaries regarding control systems with saturated stabilizing state feedback are presented in Section 2. The main results are established in Section 3, and the proofs are given in Section 4 and Section 5. Section 6 offers some examples to illustrate the main results, while Subsection 6.3 gives the steps of finding the boundary of the attraction basin. A brief conclusion is given in Section 7.

2. Preliminaries

A matrix is called to be Hurwitz, if all of its eigenvalues have negative real parts. A matrix is called to be anti-stable, if all of its eigenvalues have positive real parts.

Consider the piecewise smooth system of the following form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n},\ \|\mathbf{u}\| = \max\{|u_i|\} \le M$, where $\mathbf{u} = (u_1, \cdots, u_m)^T \in \mathbb{R}^m,\ M > 0$. Define sat: $\mathbb{R} \to \mathbb{R}$ as

$$\mathrm{sat}(s) = \mathrm{sign}(s)\min\{M,\ |s|\},$$

and for $\mathbf{u} \in \mathbb{R}^m$,

$$\mathrm{sat}(\mathbf{u}) = (\mathrm{sat}(u_1),\ \cdots,\ \mathrm{sat}(u_m))^T,$$

where $(\cdot)^T$ means the transpose. The feedback law $\mathbf{u} = \mathrm{sat}(\mathbf{K}\mathbf{x})$ is said to be stabilizing, if $\mathbf{A} + \mathbf{B}\mathbf{K}$ is Hurwitz. Assume that the system $(\mathbf{A}, \mathbf{B})$ is controllable.