

# The Exact Limits and Improved Decay Estimates for All Order Derivatives of the Global Weak Solutions to a Two-Dimensional Incompressible Dissipative Quasi-Geostrophic Equation

Linghai Zhang<sup>1,†</sup>

**Abstract** We will accomplish the exact limits for all order derivatives of the global weak solutions to a two-dimensional incompressible dissipative quasi-geostrophic equation. We will also establish the improved decay estimates with sharp rates for all order derivatives. We will consider two cases for the initial function and the external force and prove the optimal results for both cases. We will couple together existing ideas (including the Fourier transformation and its properties, Parseval's identity, iteration technique, Lebesgue's dominated convergence theorem, Gagliardo-Nirenberg-Sobolev interpolation inequality, squeeze theorem, Cauchy-Schwartz's inequality, etc) existing results (the existence of global weak solutions, the existence of local smooth solution on  $(T, \infty)$  and the elementary decay estimate with a sharp rate) and a few novel ideas to obtain the main results.

**Keywords** Incompressible dissipative quasi-geostrophic equation, All order derivatives of global weak solution, Primary decay estimates, exact limits, Improved decay estimates with sharp rates.

**MSC(2010)** 35Q20.

---

<sup>†</sup>The corresponding author.  
Email: [liz5@lehigh.edu](mailto:liz5@lehigh.edu)

<sup>1</sup>Department of Mathematics, Lehigh University, 17 Memorial Drive East,  
Bethlehem, PA USA 18015

---

## Outline of the Paper

### **1 Introduction**

- 1.1 The mathematical model equations and known related results
- 1.2 The main motivations - the main purposes - the main difficulties - the main strategies - the main advances
- 1.3 The mathematical assumptions
- 1.4 The main results

### **2 The Mathematical Analysis and the Proofs of the Main Results**

- 2.1 The elementary estimates: Case 1
- 2.2 The elementary estimates: Case 2
- 2.3 The comprehensive analysis: Case 1
- 2.4 The comprehensive analysis: Case 2
- 2.5 The primary decay estimates with sharp rates
- 2.6 The fundamental limits
- 2.7 The exact limits
- 2.8 The improved decay estimates with sharp rates

### **3 Conclusion and Remarks**

- 3.1 Summary
- 3.2 Remarks
- 3.3 Open problems
- 3.4 Technical lemmas