Journal of Nonlinear Modeling and Analysis
Volume 4, Number 4, December 2022, 753–763 http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.753

Polynomial Homogeneous Maps and Their Periods*

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Abstract We study the set of periods of the homogeneous polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{C}^n \to \mathbb{C}^n$ of degree $m > 1$. For these complex maps, we also describe the number of invariant straight lines through the origin by f^k for $k = 1, 2, \dots$ and the dynamics of f^k over them.

Keywords Homogeneous polynomial map, Period, Set of period, Self-map of a sphere.

MSC(2010) 37F10, 37C25.

1. Introduction and statement of the main results

We consider discrete dynamical systems given by a real or complex homogeneous polynomial map defined in $\mathbb{R}^n$ or $\mathbb{C}^n$ respectively. For discrete dynamical systems, the periodic orbits play an important role in understanding their dynamics. Perhaps, the best known example in this direction are the results contained in the paper entitled *"Period three implies chaos"* for continuous self-maps on the interval (see [21] or the book [2]).

The real homogeneous polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^n$ have been studied by many authors (see, for instance, the survey of Aliashvili [1] and the references quoted therein). However, not too much attention has been paid to the study of their periodic orbits with the exception of their fixed points (see, for instance, [8, 19, 27]).

Let $\mathbb{CP}^n$ be the complex projective space of dimension n. The complex homogeneous polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$ also has been considered in [1] and the references quoted there, but again not too much attention was paid to their periodic orbits. On the other hand, for the complex homogeneous polynomial maps $f : \mathbb{CP}^n \to \mathbb{CP}^n$, their set of periods have been studied (see Fornaes and Sibony [11], or [10]). On the other hand, these maps have also been studied from the point of view of their degrees (see [17]).

The study on the set of periods of the real homogeneous polynomial maps $f : \mathbb{R}^n \to \mathbb{R}^n$ is our objective for such maps, while for complex homogeneous polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$, our main goal is to study their invariant straight lines through

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*The first author was partially supported by the Agencia Estatal de Investigación (Grant No. PID2019-104658GB-I00) and the H2020 European Research Council (Grant No. MSCA-RISE-2017-777911).

the origin of coordinates by f^k for $k = 1, 2, \ldots$ and the dynamics of the map f^k restricted to these straight lines.

Let $\mathbb{F}$ be the set of all real $\mathbb{R}$ or complex $\mathbb{C}$ numbers, and let $\mathbb{F}[x_1, \ldots, x_n]$ be the ring of all polynomials in the n variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{F}$. A polynomial $P(x_1, \ldots, x_n)$ of degree m is homogeneous, if

$$P(\lambda x_1, \ldots, \lambda x_n) = \lambda^m (x_1, \ldots, x_n) \text{ for all } \lambda \in \mathbb{F} \setminus \{0\}.$$

A homogeneous polynomial map $f : \mathbb{F}^n \to \mathbb{F}^n$ of degree m is a map $f = (f_1, \ldots, f_n)$, where $f_i \in \mathbb{F}[x_1, \ldots, x_n]$ is a homogeneous polynomial of degree m for all $i = 1, \ldots, n$.

Here, f^k denotes the composition of the map f with itself k times. A point $\mathbf{x} \in \mathbb{F}^n$ is fixed by the map f, if $f(\mathbf{x}) = \mathbf{x}$. Let $k > 1$ be a positive integer. A point $\mathbf{x} \in \mathbb{F}^n$ is k–periodic or periodic point of period k by the map f, if $f^k(\mathbf{x}) = \mathbf{x}$ and $f^j(\mathbf{x}) \neq \mathbf{x}$ for $j = 1, \ldots, k-1$. The set

$$\{\mathbf{x}, f(\mathbf{x}), f^2(\mathbf{x}), \ldots, f^{k-1}(\mathbf{x})\}$$

is the *periodic orbit of* $\mathbf{x}$.

We say that the fixed points of f have period 1. We shall denote by $\mathrm{Per}(f)$, the set of periods of all periodic points of the map f. Clearly, $\mathrm{Per}(f)$ is a subset of the set $\mathbb{N}$ of all positive integers.

The sets $\mathrm{Per}(f)$ when f is a homogeneous polynomial map of degree m change completely, if the homogeneous polynomial map is real or complex. If the degree $m = 1$, then the map is linear, and its dynamics is easy to study. Here, we only consider homogeneous polynomial maps of degree $m > 1$.

In order to state our result for the complex homogeneous polynomial maps of degree $m > 1$, we need some preliminary notions.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a homogeneous polynomial map of degree m. For each $\mathbf{x} \in \mathbb{C}^n \setminus \{0\}$, we define the straight line $\mathcal{L}_{\mathbf{x}}$ through the origin of $\mathbb{C}^n$ as

$$\mathcal{L}_{\mathbf{x}} = \{\mu \mathbf{x} : \text{ for all } \mu \in \mathbb{C}\},$$

and we say that $\mathbf{x}$ is a director vector of $\mathcal{L}_{\mathbf{x}}$. A straight line through the origin of $\mathbb{C}^n$ with the director vector $\mathbf{x}$ is invariant by f, if $f(\mathbf{x}) = \lambda \mathbf{x}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then, $f(\mathcal{L}_{\mathbf{x}}) = \mathcal{L}_{\mathbf{x}}$.

It is clear that every k–periodic point $\mathbf{x}$ of f is on the invariant straight line $\mathcal{L}_{\mathbf{x}}$ of f^k, i.e., $f^k(\mathcal{L}_{\mathbf{x}}) = \mathcal{L}_{\mathbf{x}}$. Moreover, the set of straight lines

$$\{\mathcal{L}_{\mathbf{x}}, \mathcal{L}_{f(\mathbf{x})}, \mathcal{L}_{f^2(\mathbf{x})}, \ldots, \mathcal{L}_{f^{k-1}(\mathbf{x})}\}$$

is a k–periodic orbit of f in the set of all invariant straight lines through the origin, i.e., $f^k(\mathcal{L}_{\mathbf{x}}) = \mathcal{L}_{\mathbf{x}}$ and $f^\ell(\mathcal{L}_{\mathbf{x}}) \neq \mathcal{L}_{\mathbf{x}}$ for $\ell = 1, \ldots, k-1$.

The Möbius function $\mu : \mathbb{N} \to \mathbb{N}$ is defined as

$$\mu(r) = \begin{cases} 1, & \text{if } r = 1, \\ 0, & \text{if some } k^2 | r \text{ for some } k \in \mathbb{N}, \\ (-1)^s, & \text{if } r = p_1 \cdots p_s \text{ (distinct primes)}. \end{cases}$$

Theorem 1.1. *Let* $f : \mathbb{C}^n \to \mathbb{C}^n$ *be a homogeneous polynomial map of degree* $m > 1$ *such that* $f(\mathbf{x}) \neq 0$, *if* $\mathbf{x} \neq 0$. *Let* k *be a positive integer.*