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Poincaré Bifurcation from an Elliptic Hamiltonian of Degree Four with Two-saddle Cycle*

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Abstract In this paper, we consider Poincaré bifurcation from an elliptic Hamiltonian of degree four with two-saddle cycle. Based on the Chebyshev criterion, not only one case in the Liénard equations of type $(3, 2)$ is discussed again in a different way from the previous ones, but also its two extended cases are investigated, where the perturbations are given respectively by adding $\varepsilon y(d_0 + d_2 v^{2n})\frac{\partial}{\partial y}$ with $n \in \mathbb{N}^+$ and $\varepsilon y(d_0 + d_4 v^4 + d_2 v^{2n+4})\frac{\partial}{\partial y}$ with $n = -1$ or $n \in \mathbb{N}^+$, for small $\varepsilon > 0$. For the above cases, we obtain all the sharp upper bound of the number of zeros for Abelian integrals, from which the existence of limit cycles at most via the first-order Melnikov functions is determined. Finally, one example of double limit cycles for the latter case is given.

Keywords Perturbed Hamiltonian system, Poincaré bifurcation, Abelian integral, Chebyshev criterion.

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1. Introduction

This paper deals with Liénard equations in the following form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}t} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = P(v) + yQ(v), \end{cases} \tag{1.1}$$

where P and Q are two polynomials in the variable v, and if deg $P = m$ and deg $Q = n$, equations (1.1) are called Liénard system of type (m, n). When it comes to the famous 16th problem of Hilbert, equations (1.1) have been studied extensively on asking for an upper bound on the number of limit cycles [18, 19], especially in respect to the weak 16th problem of Hilbert [1]. For studying the limit cycles in certain vector fields with specific degree for (1.1), we construct the perturbed Hamiltonian system by setting $Q(v) = \varepsilon g(v)$ with $\varepsilon > 0$, but it is small as follows

$$\frac{\mathrm{d}v}{\mathrm{d}t} = H_y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -H_v + \varepsilon g(v)y, \tag{1.2}$$

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where $H(v,y) = \frac{y^2}{2} + \int P(v)\mathrm{d}v$ is a Hamiltonian polynomial function of degree $m+1$. Thus, we can investigate Poincaré bifurcation of limit cycles via calculating the zeros of the elliptic integrals, i.e., Abelian integrals obtained by integrating the related 1-form $yq(v)\mathrm{d}v$ over the compact level curves of the Hamiltonian H.

There are some effective research methods and many good results which are obtained for the zeros of Abelian integrals and Poincaré bifurcation of system (1.2) (see the review literature [16], the recent papers [2, 10, 25] and references therein). As for the existence, uniqueness and number of limit cycles, it is worth noting that Jiang and Han [15] extended the investigation from the continuous nonlinear Liénard-type differential systems (1.1) to the discontinuous ones, and discussed the qualitative properties of the crossing limit cycles.

When $m+n \le 4$, type (m,n) of the Liénard equations (1.1), except for type $(1,3)$, has been given a complete study, and the corresponding Abel integrals were also proved to have one at most zero [3, 16]. As for its type (m,n) with $m+n=5$, the comprehensive and detailed studies on type $(3,2)$ have been carried out by Dumortier and Li in a series of papers [4–7], and the corresponding supremum of the zero number of Abel integral was obtained respectively for all the five different cases.

We know that type $(3,2)$ corresponds to small perturbations of Hamiltonian vector fields with an elliptic Hamiltonian of degree four, and the perturbations are given by adding $\varepsilon y(x^2+\beta x+\alpha)\frac{\partial}{\partial y}$ for small $\varepsilon > 0$. It is worth mentioning that, for Two-saddle Cycle Case (A) and Saddle Loop Case (B) presented in [4], after linear rescaling their Hamiltonians are given by the following functions

$$H(v,y) = \frac{y^2}{2} - \frac{1}{4}v^4 - \frac{(\lambda-1)}{3}v^3 + \frac{\lambda}{2}v^2, \tag{1.3}$$

where $\lambda \ge 1$, and $\lambda = 1$ corresponds to Case (A), while $\lambda > 1$ corresponds to Case (B). Case (A) is rather simple as the limiting situation of Case (B), and its Abel integral was proved to have one at most zero in [14] and [4] respectively.

Here, we shall develop a utilized approach based on Chebyshev criterion [9, 21] to prove the above conclusion once again, and we shall investigate the extended Case (A) more, namely for $\lambda = 1$ with the following more generic form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}\xi} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = (v+1)v(v-1) + \varepsilon g(v)y, \end{cases} \tag{1.4}$$

where $g(v) = \sum_{i=0}^{2m} d_i v^i$, $m \in \mathbb{N}^+, d_i \in \mathbb{R}$. Obviously, the $g(v)$ in the perturbed part of system (1.4) is no longer restricted to quadratic polynomial, and the following two classes of $g(v)$ will be discussed here

$$\begin{aligned} &g_a(v) = d_0 + d_{2n}v^{2n},\ n = 1, 2, \cdots, \\ &g_b(v) = d_0 + d_4 v^4 + d_{2n+4}v^{2n+4},\ n = -1, 1, 2, \cdots. \end{aligned} \tag{1.5}$$

The Picard-Focus equations and Riccati equations in algebraic geometry were applied as the important research approaches in [4], which are still often used (see, for example, [13, 17]). Here, by using the method of Chebyshev criterion presented in [9, 21], and via strict proof of symbolic calculation, we determine the zero-point number of the Abelian integrals, i.e., the first-order Melnikov functions. Many