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Existence and Uniqueness Theorems for a Three-step Newton-type Method under L-Average Conditions

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Abstract In this paper, we study the local convergence of a three-step Newton-type method for solving nonlinear equations in Banach spaces under weaker hypothesis. More precisely, we derive the existence and uniqueness theorems, when the first-order derivative of nonlinear operator satisfies the L-average conditions instead of the usual Lipschitz condition, which have been discussed in the earlier study.

Keywords Banach space, Nonlinear equation, Lipschitz condition, L-average, Convergence ball.

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1. Introduction

Let $T : \mathfrak{D} \subseteq M \to N$ be a nonlinear operator from a Banach space M to another Banach space N, where $\mathfrak{D}$ is a non-empty open convex subset, and T is Fréchet differentiable nonlinear operator. Nonlinear equations arise in many fields of science and engineering, and are defined by

$$T(\alpha) = 0. \tag{1.1}$$

The well-known iterative method for finding the solution of above type equation is Newton's method, which is given by

$$\alpha_{n+1} = \alpha_n - [T'(\alpha_n)]^{-1}T(\alpha_n), \ n \geq 0. \tag{1.2}$$

Newton's method [8] is a well established iterative method, which converges quadratically. It was first discussed by Kantorovich [6], and then investigated by Rall [10]. Some higher-order methods that do not require the computation of second-order derivatives have been developed in [4, 5, 7, 9] and other literature. Due to high operational cost, the methods of higher R-order convergence are not normally used despite fast speed of convergence. However, the methods of higher R-order are useful in the problems of stiff system [6], where fast convergence is required.

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Here, we discuss the local convergence of a three-step Newton-type method under the L-average conditions, which is expressed as

$$\begin{aligned}\beta_n &= \alpha_n - [T'(\alpha_n)]^{-1}T(\alpha_n),\\ \gamma_n &= \alpha_n - [T'(\alpha_n)]^{-1}[T(\alpha_n)+T(\beta_n)],\\ \alpha_{n+1} &= \alpha_n - [T'(\alpha_n)]^{-1}[T(\alpha_n)+T(\beta_n)+T(\gamma_n)],\ n\geq 0.\end{aligned} \tag{1.3}$$

Method (1.3) is characterized by the simplest fourth-order iterative method, which is not involved in the second derivative. The local convergence of this method has been studied by Argyros et al., [3] under Lipschitz and center Lipschitz conditions, which are given by

$$||[T'(\alpha^*)]^{-1}(T'(\alpha)-T'(\beta))|| \leq L||\alpha-\beta||, \forall\ \alpha,\beta\in B(\alpha^*,r) \tag{1.4}$$

and

$$||[T'(\alpha^*)]^{-1}(T'(\alpha)-T'(\alpha^*))|| \leq L_0||\alpha-\alpha^*||, \forall\ \alpha\in B(\alpha^*,r). \tag{1.5}$$

For some kinds of domain with center such as $B(\alpha^*,r)$, sometimes it is not necessary for the inequality to hold for any α,β in the domain, and it is only required to hold for any α and for points lying on the connecting line $\alpha^\tau=\alpha^*+\tau(\alpha-\alpha^*)$ between α and α^*, where $0\leq\tau\leq 1$. Recently, in [11], Wang has introduced the Lipschitz condition and center Lipschitz condition with L-average, which are given by for $[T'(\alpha^*)]^{-1}T'(\alpha)$:

$$||[T'(\alpha^*)]^{-1}(T'(\alpha)-T'(\alpha^\tau))|| \leq \int_{\tau\kappa(\alpha)}^{\kappa(\alpha)} L(u)du, \forall\ \alpha,\alpha^\tau\in B(\alpha^*,r), 0\leq\tau\leq 1 \tag{1.6}$$

and

$$||[T'(\alpha^*)]^{-1}(T'(\alpha)-T'(\alpha^*))|| \leq \int_{0}^{\kappa(\alpha)} L_0(u)du, \forall\ \alpha\in B(\alpha^*,r), \tag{1.7}$$

where L and L_0 are positive integrable function instead of constant. If L and L_0 are constants, then R. H. S. of the above two equations becomes $L||\alpha-\alpha^\tau||$ and $L_0||\alpha-\alpha^*||$.

Encouraged by the development discussed above, in this paper, we first mention the definitions of Lipschitz condition and center Lipschitz condition with L-average for algorithm (1.3), and some theorems are then derived. The first L-average conditions have been used to study the local convergence without additional hypotheses, along with an error estimate. In the second theorem, the domain of uniqueness of solution has been derived under center Lipschitz condition. Also, few corollaries are stated.

The rest parts of this paper are organized as follows. Section 2 includes the definitions related to L-average conditions. The local convergence and its domain of uniqueness are respectively mentioned in Section 3 and Section 4.