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# The Exact Solutions for the Benjamin-Bona-Mahony Equation*

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**Abstract** The Benjamin-Bona-Mahony (BBM) equation represents the unidirectional propagation of nonlinear dispersive long waves, which has a clear physical background, and is a more suitable mathematical and physical equation than the KdV equation. Therefore, the research on the BBM equation is very important. In this article, we put forward an effective algorithm, the modified hyperbolic function expanding method, to build the solutions of the BBM equation. We, by utilizing the modified hyperbolic function expanding method, obtain the traveling wave solutions of the BBM equation. When the parameters are taken as special values, the solitary waves are also derived from the traveling waves. The traveling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. The modified hyperbolic function expanding method is direct, concise, elementary and effective, and can be used for many other nonlinear partial differential equations.

**Keywords** Generalized hyperbolic tangent function method, The modified hyperbolic function expanding method, Traveling wave solution, Balance coefficient method, Partial differential equation.

**MSC(2010)** 35A08, 35A24, 35C07, 35C08, 35G20.

## 1. Introduction

Many phenomena in natural science can be described by nonlinear partial differential equations (NLPDE), for instance, physics, ecology, medicine, zoology, fiber communications, fluid dynamics, propagation of waves, marine engineering, plasma physics, incompressible fluid, ocean and rogue waves, photonics, optics, optical-fiber communications, superconductors, arterial mechanics and cosmic plasmas [1, 2, 4–7, 13, 17, 19, 22, 23, 26–29, 44]. Due to the wide application of NLPDE in real life, many scholars want to study their application mechanism by solving exact solutions of NLPDE, paving way for the in-depth research [44] in the future.

So far, we have generally obtained the exact solutions of NLPDE by using mathematical software such as Maple and Mathematica [3, 28, 30, 44, 45]. Common meth-

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ods to solving NLPDE include the tanh method [18, 24, 38, 47], the Hirota's bilinear method [40], the sine-cosine method [33], the Exp-function method [10, 16], the inverse scattering method [20], the $\frac{G'}{G}$-expansion method [25, 36], the $sn$-$ns$ method [15], the Darboux transformation method [9], the $F$-expansion method [21], the polynomial expansion method [22], the modified polynomial expansion method [23], the Bäcklund transformation [35], etc. Specifically, using the Hirota's bilinear method and the tanh-coth method, in [39], Wazwaz yielded the $N$-soliton solutions, $N = 1, 2, 3, 4, 5$, for the Kadomtsev-Petviashvili equation. In [40], the author obtained the single-soliton solution and the $N$-soliton solution about the Sawada-Kotera-Kadomtsev-Petviashvili equation. In [37], the author obtained the traveling wave solutions to the KdV equation, the generalized KdV equation, the $K(n, n)$ equations, the Boussinesq equation, the RLW equation, the BBM equation and the Phi-four equation via the sine-cosine method. In [49], Zhang and Huang acquired $N$-soliton solutions to the KdV equation with the variable coefficients Exp-function method. In [1], Abdel-Gawad and Osman obtained a wide class of exact solutions to the variable coefficient KdV equation by using the extended unified method, and presented a new algorithm for treating the coupled NLPDE. In [34], Vakhnenko and Parkes obtained the exact $N$-soliton solutions to the Vakhnenko equation utilizing the inverse scattering method. In [41], the author yielded the traveling wave solutions to the Zhiber-Shabat equation, the Liouville equation, the sinh-Gordon equation, the Dodd-Bullough-Mikhailov equation and the Tzitzeica-Dodd-Bullough equation via the tanh method and the extended tanh method, which had showed their different physical structures. Employing the modified extended tanh method, in [31], Raslan, Ali and Shallal acquired the traveling wave solutions for the space-time fractional nonlinear partial differential equations, for instance, the fractional equal width wave equation and the fractional modified equal width wave equation. In [46], Zahran and Khater obtained the traveling wave solutions to the Bogoyavlenskii equation, which had showed broad applicability. In [12], by using the first integral method, the tanh-coth method, the sech-csch method, the tan-cot method and the sec-csc ansatz, Darvishi et al., acquired the traveling wave solutions to the $(2 + 1)$-dimensional Zakharov-like soliton equation. In [4], Akbar, Kayum and Osman obtained special solutions, including the bright solitons, the periodic solutions, the compaction solutions, the bell-shape solutions, the parabolic shape solutions, the singular periodic solutions, the plane shape solutions and some new types of solitons, for the $(3 + 1)$-dimensional Zakharov-Kuznetsov (ZK) and the new extended quantum ZK equations by using the enhanced modified simple equation method. In [32], the exact traveling wave solutions to the $M$-fractional generalized reaction Duffing model and the density dependent $M$-fractional diffusion reaction equation by using three fertile methods, including the $(\frac{G'}{G}, \frac{1}{G})$ method, the modified $\frac{G'}{G^2}$ method and the $\frac{1}{G'}$-expansion method. In [36], Wang, Li and Zhang yielded solitary wave solutions and periodic wave solutions for the BBM equation and the modified Benjamin-Bona-Mahony equation by employing a generalized $\frac{G'}{G}$-expansion method. In [14], Ghanbari, Baleanu and Qurashi obtained some new exact optical solitons for the generalized Benjamin-Bona-Mahony equation via the generalized exponential rational function method, and detailed the physical interpretation of these solutions. In [43], Yan et al., obtained non-local symmetry and