Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 4, Number 4, December 2022, 615–627 DOI:10.12150/jnma.2022.615

Rational Solutions to the KdV Equation in Terms of Particular Polynomials

Pierre Gaillard[1,†]

Abstract Here, we construct rational solutions to the KdV equation by particular polynomials. We get the solutions in terms of determinants of the order n for any positive integer n, and we call these solutions, solutions of the order n. Therefore, we obtain a very efficient method to get rational solutions to the KdV equation, and we can construct explicit solutions very easily. In the following, we present some solutions until order 10.

Keywords Polynomial, Bilinear differential operator, Rational solution.

MSC(2010) 35C99, 35Q35, 35Q53.

1. Introduction

We consider the KdV equation in the following normalization

$$4u_t = 6uu_x + u_{xxx}. \tag{1.1}$$

As usual, the subscripts x and t denote partial derivatives. This equation appeared in the footnote on Page 360 of Boussinesq's massive 680-page memoir [4] written in 1872. Korteweg and Vries [10] studied equation (1.1) in a paper published in 1895, and from then on, this equation had carried their names. This equation has described the propagation of waves with weak dispersion in various nonlinear media. Gardner et al., [7] proposed a method of resolution in 1967. In 1971, Zakharov and Faddeev [19] proved this equation as a complete integrable system. In the same year, Hirota [8] constructed the solutions by using the bilinear method, and lot of works came into existence in the following years. We can mention, for example, Its and Matveev [9] in 1975, Lax [12] in the same year, Airault et al., [3] in 1977, Adler and Moser [2] in 1978, Ablowitz and Cornille [1] in 1979, Freeman and Nimmo [5] in 1984, Matveev [17] in 1992, Ma [16] in 2004, Kovalyov [11] in 2005 and Ma [14] in 2015.

In the following, we consider certain polynomials and construct rational solutions using determinants of the order n. The proof of the result is based on the verification of the corresponding Hirota bilinear expression for the KdV equation.

Therefore, we get a very efficient method to construct rational solutions to the KdV equation. We give explicit solutions in the simplest cases for orders $n = 1$ until 10.

[†]the corresponding author.
Email address: pierre.gaillard@u-bourgogne.fr (P. Gaillard)
[1]Institut de Mathématiques, Université de Bourgogne-Franche Comté, Dijon, 21000, France

2. Rational solutions to the KdV equation

We consider the polynomial p_n defined by

$$\begin{cases} p_{3k}(x,t)=\sum_{l=0}^{k}\dfrac{x^{3l}}{(3l)!}\dfrac{(t)^{k-l}}{(k-l)!}, \quad k\geq 0,\\ p_{3k+1}(x,t)=\sum_{l=0}^{k}\dfrac{x^{3l+1}}{(3l+1)!}\dfrac{(t)^{k-l}}{(k-l)!}, \quad k\geq 0,\\ p_{3k+2}(x,t)=\sum_{l=0}^{k}\dfrac{x^{3l+2}}{(3l+2)!}\dfrac{(t)^{k-l}}{(k-l)!}, \quad k\geq 0,\\ p_n(x,t)=0, \quad n<0. \end{cases} \tag{2.1}$$

We denote $A_n(x,t)$ the following determinant

$$A_n(x,t)=\det(p_{n+1-2i+j}(x,t))_{\{1\leq i\leq n,\,1\leq j\leq n\}}. \tag{2.2}$$

With these notations, we have the following result.

Theorem 2.1. *The function $v_n(x,t)$ defined by*

$$v_n(x,t)=2\partial_x^2(\ln(A_n(x,t))) \tag{2.3}$$

is a rational solution to the (KdV) equation (1.1)

$$4u_t=6uu_x+u_{xxx}. \tag{2.4}$$

Proof. We know that $v_n(x,t)=2\partial_x^2(\ln f(x,t))$ is a solution to the KdV equation, if f satisfies the following equation

$$(D_x^4-4D_xD_t)f\cdot f=0, \tag{2.5}$$

where D is the bilinear differential operator.

We have to verify (2.5) for $f=\det(p_{n+1-2i+j}(x,t))_{\{1\leq i\leq n,\,1\leq j\leq n\}}$. We denote by C_j the following column, $1\leq j\leq n$

$$C_j=\begin{pmatrix} p_{n-1+j}\\ p_{n-3+j}\\ \vdots\\ p_{-n+1+j}\end{pmatrix}. \tag{2.6}$$

With these notations, $A_n(x,t)$ can be written as $A_n(x,t)=|C_1,\dots,C_n|$.

We denote H as the expression $H=(D_x^4-4D_xD_t)f\cdot f$. We have to evaluate H. The polynomials p_k verify $\partial_x(p_k)=p_{k-1}$ and $\partial_t(p_k)=p_{k-3}$.

Therefore, H can be written as

$$\begin{aligned} H=&[|C_{-3},C_2,C_3,C_4,C_5,\dots,C_n|+3|C_{-2},C_1,C_3,C_4,C_5,\dots,C_n|\\ &+2|C_{-1},C_0,C_3,C_4,C_5,\dots,C_n||C_{-1},C_1,C_2,C_4,C_5,\dots,C_n|\\ &+|C_0,C_1,C_2,C_3,C_5,\dots,C_n|]\times|C_1,C_2,C_3,C_4,C_5,\dots,C_n|\\ &-4[|C_{-2},C_2,C_3,C_4,C_5,\dots,C_n|+2|C_{-1},C_1,C_3,C_4,C_5,\dots,C_n| \end{aligned}$$