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A Fast Cartesian Grid-Based Integral Equation Method for Unbounded Interface Problems with Non-Homogeneous Source Terms

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Abstract. This work presents a fast Cartesian grid-based integral equation method for unbounded interface problems with non-homogeneous source terms. The unbounded interface problem is solved with boundary integral equation methods such that infinite boundary conditions are satisfied naturally. This work overcomes two difficulties. The first difficulty is the evaluation of singular integrals. Boundary and volume integrals are transformed into equivalent but much simpler bounded interface problems on rectangular domains, which are solved with FFT-based finite difference solvers. The second one is the expensive computational cost for volume integrals. Despite the use of efficient interface problem solvers, the evaluation for volume integrals is still expensive due to the evaluation of boundary conditions for the simple interface problem. The problem is alleviated by introducing an auxiliary circle as a bridge to indirectly evaluate boundary conditions. Since solving boundary integral equations on a circular boundary is so accurate, one only needs to select a fixed number of points for the discretization of the circle to reduce the computational cost. Numerical examples are presented to demonstrate the efficiency and the second-order accuracy of the proposed numerical method.

AMS subject classifications: 35J05, 65N06, 65N38

Key words: Interface problem, unbounded domain, boundary integral equation, kernel-free method, auxiliary circle, Cartesian grid method, fast algorithm.

1 Introduction

Elliptic partial differential equations (PDEs) on unbounded domains arise in many physics and engineering problems, including solid and fluid mechanics [29, 33], electromagnetic

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and mechanical wave propagation [12, 17], molecular dynamics [25] and geophysics [9]. In numerical computations, the unboundedness of the domain complicates the effective numerical solution of the problems.

Over the past decades, a large number of numerical methods for PDEs on unbounded domains have been developed. One frequently used approach is to truncate the unbounded domain and impose appropriate boundary conditions, including Dirichlet-to-Neumann (DtN) artificial boundary conditions [26–28], radiation boundary conditions [2, 16], absorbing boundary conditions [1], perfectly matched layers [4, 5], and perfect absorbing layers [40]. One of the other well-established methods is the spectral method. Among the many spectral approaches, one class employs a transform that maps a bounded domain to an unbounded domain and then uses mapped orthogonal functions [7, 15, 32]. Some others choose orthogonal basis in unbounded domains, including Hermite and Laguerre functions [30, 37, 38]. We recommend a review article [31].

Boundary integral methods (BIMs) have been used extensively in elliptic PDEs because of their ability to handle complex geometry and unbounded domains [21]. BIMs use boundary integrals to represent solutions. For PDEs on unbounded domains, the far-field condition is incorporated into the representation itself, and they are transformed into integral equations on the boundary. It reduces space dimensions of the problem by one and achieves optimal computational complexity when combined with fast algorithms. For non-homogeneous PDEs, there is a volume integral term. It brings large computational complexity which weakens the advantage of dimension reduction. Two major difficulties of solving BIMs are singular and nearly singular integrals evaluation and the high time complexity for numerical calculation.

Some methods can overcome the difficulty of singular and nearly singular integrals. One class of methods are based on quadrature schemes, including [3, 8, 19]. They achieve high accuracy by making corrections or adding quadrature points. Another class of methods evaluates singular integrals by solving PDEs [6, 42]. The integral boundary is embedded into a rectangle, and the integral can be represented by the solution to an equation on the rectangular domain.

The kernel-free boundary integral method [41–44] belongs to the second class. The singular or nearly singular boundary or volume integral evaluation requires solving an equivalent but much simpler interface problem on a rectangle. The rectangular domain is discretized with a uniform Cartesian grid and the operator is approximated by the finite difference scheme. Since Boundary and volume integrals or their derivatives are discontinuous, the finite difference scheme loses accuracy when its stencils across the integral curve. These inaccurate parts are modified by corrections, which are calculated with density functions of boundary and volume integrals. To keep the finite difference matrix unchanged, these corrections are added to the right-hand side. For equations on Cartesian grids, fast and high-accuracy solvers are readily available, such as FFT-based solvers and geometry multigrid methods.

This work proposes a fast and accurate method to evaluate boundary and volume integrals for elliptic PDEs on two-dimensional unbounded domains. Enlighten by the